INTRODUCTION TO LINEAR ALGEBRA

by
IRENE FARKAS
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Intending their book primarily as a first-year course for physics and engineering students, rather than for mathematicians, the authors proceed throughout from the concrete and special to the abstract and general. Thereby, they supply their intended readers with a text which will be found far more readily comprehensible than a purely theoretical treatise, even though they seek and attain strict rigour in their presentation and proof of the theorems.

The book is liberally provided with exercises, either with hints for their solution or with answers.

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We dedicate this book
to the memory of Dénes Wolf

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PREFACE

This book is intended to be an introduction in the genuine sense of the word. It is written for first year students majoring in mathematics, science, or engineering who have no background beyond their high-school knowledge. It can be used as a textbook for a one-semester course of linear algebra running parallel to a course of mathematical analysis in which real numbers are treated.

There are several fine books on linear algebra which begin with the axioms defining groups, fields, linear spaces, etc., and continue by proving the main theorems of these algebraic structures. There seemed no need for another text of the same kind, but we thought that the approach leading the beginner to the axioms themselves was much neglected. The aim of the lectures of the second author, and now of this book, is to present the development of linear algebra from experience–that is, to bridge the concrete structures (partly taught in high school) and the axiomatic treatment of the subject. With this aim we embarked on the dangerous path of applying intuitive concepts at the beginning of a rigorous text.

Accordingly, in Chapter I we deal with common vector algebra in the ordinary space. Here vectors are models of the intuitive concept of displacement. Apart from this, the treatment is rigorous from the start. The subject is arranged as a model for the abstract ideas that are introduced later. Similar reasons explain the inclusion of complex numbers in Chapter II, with emphasis on the linear space structure in the complex field. A fairly comprehensive elementary matrix algebra is given in Chapter III including a straightforward theory of determinants. One more concrete model of linear space is treated here: the set of ordered n-tuples. (The authors believe that this particular structure must be studied before the abstract treatment of linear spaces, to make coordinate representation of vectors possible without confusion.) Chapter IV deals with the theory of systems of linear equations. A survey of the structure of the solution set is given here and we hope that this will be of use to the reader studying, for instance, linear differential and integral equations. The axiomatic definition of the linear space is given in Chapter V. The concept and the properties of the Euclidean space are also treated here. Chapter VI concerns linear operators mapping the linear or the Euclidean space into itself, with emphasis on symmetric (self-adjoint) operators. Linear dependence and independence are considered the basic concepts of linear algebra and play a leading part throughout the different topics treated in the book.

The text is divided into Sections and these are numbered independently of the Chapters. In each Section the Definitions, the Theorems (Lemmas and Corollaries used essentially synonymously), the Examples, the Formulae, and the Figures are numbered separately. The Examples form an integral part of the theory. Exercises have been included at the end of each Chapter and hints, answers or both are given to each Exercise at the end of the volume. The end of a proof is denoted by a dot and an exclamation mark: . ! Standard elementary mathematical notation is used otherwise. For example, $a \in A$ means that a is an element of the set A, and $A \subset B$ means that A is a subset of B.

We wish to express our gratitude and appreciation for the valuable criticism and suggestions given by Professor P. Rózsa who read the text with great care.

Budapest, May 1974

I. F.
M. F.

CHAPTER ONE

ELEMENTARY VECTOR ALGEBRA

1. Vectors in Ordinary Space. Elementary Operations

Many of the quantities that occur in the laws of mechanics or physics cannot be characterized merely by a single real number. Displacement, velocity, force, etc., are unambiguous only if their direction is given as well as their magnitude. Displacement is, for example, an operation by which an arbitrary point of space moves to another point. To characterize a displacement $\mathbf{x}$ its 'magnitude' (a non-negative number), its 'direction' (represented, say, by a straight line parallel to the displacement) and its 'sense' (one of the two possible senses of the displacement along the line) must be given. Two arbitrary points of space, A and B, given in definite order define exactly one displacement $\mathbf{x}$ which carries A into the position B. This displacement can be accomplished, then, on any point of space. Conversely, an arbitrarily given point A and a displacement $\mathbf{x}$ define exactly a point B into which A is carried over by $\mathbf{x}$. Two displacements carried over one after the other can be performed by a single displacement called the sum of the former two. We are going to talk about vectors keeping in mind the intuitive concept of displacement.

We say that a *vector* $\mathbf{x}$ is determined by its *length (magnitude, absolute value)* $|\mathbf{x}|$, its *direction*, and its *sense*, i.e. a non-negative number, a straight line (representing all the straight lines parallel to the given one), and the sense in which the given straight line is directed. Two vectors are *equal* (are the same) if their lengths, directions, and senses are equal. The vector $\mathbf{x}$ whose length is zero, $|\mathbf{x}| = 0$, is called the *zero vector* and denoted by $\mathbf{0}$. We say that the vectors $\mathbf{a}$ and $\mathbf{b}$ are *parallel (collinear)*, $\mathbf{a} \parallel \mathbf{b}$, if their directions are equal. We say that the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are *coplanar* if there is a plane parallel to all three vectors.

Any ordered pair of points (A, B) of the space define one and only one vector, denoted by $\mathbf{x} = \overrightarrow{AB}$, whose length is equal to the distance between A and B, whose direction is given by the straight line determined by A and B (if $A \neq B$), and whose sense is from A to B. The ordered pair (A, A) defines the zero vector $\mathbf{0} = \overrightarrow{AA}$. The direction and sense of the zero vector is undetermined. *The zero vector is considered to be parallel to any vector.* The vector $\mathbf{x} = \overrightarrow{AB}$ defined by the ordered pair of points (A, B) can be represented by the directed straight line segment AB. If this is done we call A the *initial point* and B the *terminal point* of the vector $\mathbf{x}$. In drawing, this directed straight line segment is represented by an arrow with its head at B.

An arbitrary point A and an arbitrary vector $\mathbf{x}$ define one and only one point B, namely the endpoint different from A of the straight line segment which we get by taking the straight line passing through A in the direction given by the vector $\mathbf{x}$ and measuring off this straight line the distance $|\mathbf{x}|$ from the point A in the sense defined by $\mathbf{x}$. Clearly, $\mathbf{x} = \overrightarrow{AB}$.

The directed straight line segment AB (which is fixed to the point A and to the line determined by A and B) may represent the vector $\overrightarrow{AB}$ but is not identical with it. The vector $\overrightarrow{AB}$ can be represented by any directed straight line segment obtained from AB by parallel displacement. If A, B, and C are arbitrary points, $\mathbf{x} = \overrightarrow{AB}$, and the point determined by C and $\mathbf{x}$ is D (i.e. $\overrightarrow{CD} = \mathbf{x}$), then, of course, $\overrightarrow{AB} = \overrightarrow{CD}$. Sometimes it is necessary to fix a vector to a given point. This is why the last identity is sometimes interpreted as if the vector $\overrightarrow{CD}$ had been obtained by parallel displacement of the vector $\overrightarrow{AB}$ from the point A to the point C. In fact, of course, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are the *same* vector.

From elementary properties of a parallelogram, it follows that

$$\text{if } \overrightarrow{AB} = \overrightarrow{CD}, \text{ then } \overrightarrow{AC} = \overrightarrow{BD} \tag{1.1}$$

(see Fig. 1.1).

Let $\mathbf{a}$ and $\mathbf{b}$ be two arbitrary vectors. The *angle α of the vectors* $\mathbf{a}$ and $\mathbf{b}$ is defined the following way. If A is an arbitrary point and $\mathbf{a} = \overrightarrow{AB}$, $\mathbf{b} = \overrightarrow{AC}$, then $\alpha = BAC \sphericalangle$. It is clear that α does not depend on the choice of A and that $0 \leq \alpha \leq \pi$. If the angle of two vectors is equal to $\pi/2$, then we say that these vectors are *orthogonal (perpendicular)*. The angle of the zero vector and any other vector is undetermined. *The zero vector is considered to be orthogonal to any vector.*

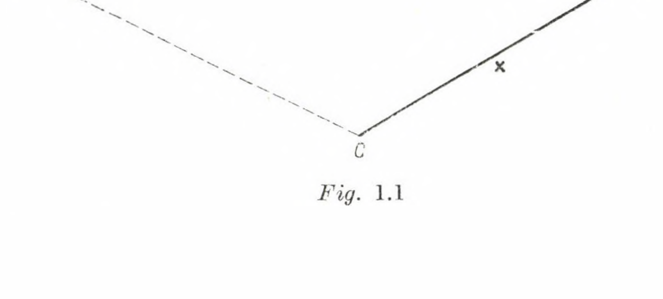


Fig. 1.1

Definition 1.1. Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors and A an arbitrary point. Define the points B and C by $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{BC} = \mathbf{b}$, respectively. The vector $\overrightarrow{AC}$ is then called the *sum of* $\mathbf{a}$ *and* $\mathbf{b}$ and is denoted by $\mathbf{a} + \mathbf{b}$.

It is to be shown that the sum of $\mathbf{a}$ and $\mathbf{b}$ is independent of the choice of A. Let A' be an arbitrary point, B' and C' defined by $\overrightarrow{A'B'} = \mathbf{a}$ and $\overrightarrow{B'C'} = \mathbf{b}$, respectively. Since $\overrightarrow{AB} = \overrightarrow{A'B'}$ and $\overrightarrow{BC} = \overrightarrow{B'C'}$, therefore by (1.1) $\overrightarrow{AA'} = \overrightarrow{BB'}$ and $\overrightarrow{BB'} = \overrightarrow{CC'}$, i.e. $\overrightarrow{AA'} = \overrightarrow{CC'}$. Using again (1.1) we have that $\overrightarrow{AC} = \overrightarrow{A'C'}$.

The terms in a sum of vectors are called the *components* of the sum and the sum is sometimes called the *resultant* of the components. Addition of vectors satisfies the following laws which are immediate consequences of Definition 1.1 and of simple geometrical facts. Addition is *commutative*, i.e. for any two vectors

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}. \tag{1.2}$$

(See Fig. 1.2 which at the same time shows the so-called *parallelogram law.*) Addition is *associative*, i.e. for any three vectors

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) = \mathbf{a} + \mathbf{b} + \mathbf{c}. \tag{1.3}$$

By repeated application of the identities (1.2) and (1.3) we find that the sum of a finite number of vectors is unaltered if the components are interchanged or grouped. A vector stays unaltered if the zero vector is added to it, i.e. for any vector $\mathbf{a}$

$$\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}. \tag{1.4}$$

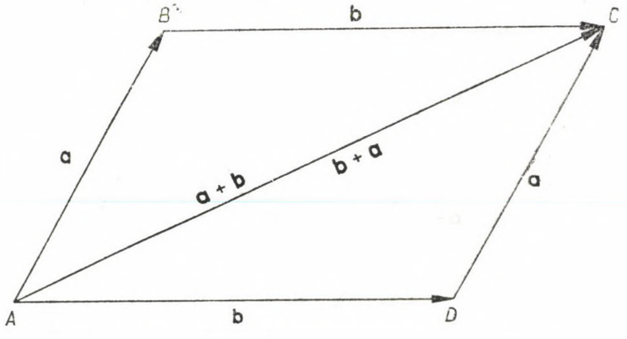


Fig. 1.2

Given a vector $\mathbf{a}$ we denote by $-\mathbf{a}$ and call the *opposite vector* of $\mathbf{a}$ the vector that satisfies

$$\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}. \tag{1.5}$$

Let A be an arbitrary point; if $\mathbf{a} = \overrightarrow{AB}$, then obviously $-\mathbf{a} = \overrightarrow{BA}$. The vectors $\mathbf{a}$ and $-\mathbf{a}$ have the same length and direction but their senses are opposite.

Definition 1.2. Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors; their *difference* is the vector defined by

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}). \tag{1.6}$$

The basic *triangle inequality*

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}| \tag{1.7}$$

follows (for arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$) immediately from Definition 1.1, since the sum of two sides of a triangle is greater than or equal to the third side (see Fig. 1.2). A simple consequence of (1.7) is the inequality

$$||\mathbf{a}| - |\mathbf{b}|| \leq |\mathbf{a} - \mathbf{b}|, \tag{1.8}$$

since for arbitrary $\mathbf{a}$ and $\mathbf{b}$

$$\mathbf{a} = (\mathbf{a} + \mathbf{b}) + (-\mathbf{b}), \ \mathbf{b} = (\mathbf{a} + \mathbf{b}) + (-\mathbf{a});$$

thus by (1.7)

$$|\mathbf{a}| \leq |\mathbf{a} + \mathbf{b}| + |\mathbf{b}|, \ |\mathbf{b}| \leq |\mathbf{a} + \mathbf{b}| + |\mathbf{a}|,$$

or

$$-|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| - |\mathbf{b}| \leq |\mathbf{a} + \mathbf{b}|$$

and the latter inequality is equivalent to (1.8).

As can be seen from Fig. 1.3, if P is an arbitrary point and $\overrightarrow{PA} = \mathbf{a}$, $\overrightarrow{PB} = \mathbf{b}$, then $\mathbf{a} - \mathbf{b} = \overrightarrow{BA}$.

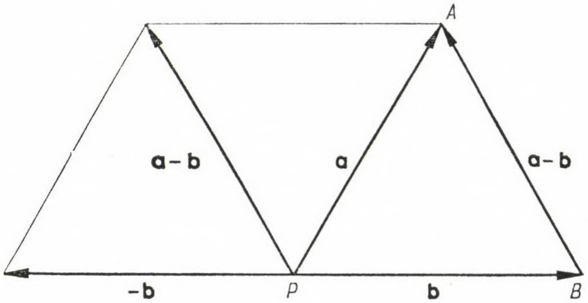


Fig. 1.3

If we want to lay emphasis on the fact that a (real) number is not a vector, we call it a *scalar*.

DEFINITION 1.3. *(Multiplication of a vector by a scalar)*. Let $\lambda > 0$ be a scalar and $\mathbf{a}$ a vector; $\lambda\mathbf{a}$ is the vector whose length is $\lambda\,|\,\mathbf{a}\,|$ and whose direction and sense are equal to those of $\mathbf{a}$. Let $\lambda < 0$ be a scalar and $\mathbf{a}$ a vector; $\lambda\mathbf{a}$ is the vector whose length is $|\,\lambda\,|\;|\,\mathbf{a}\,|$, whose direction is equal to that of $\mathbf{a}$, and whose sense is opposite to $\mathbf{a}$. Finally $0\mathbf{a} = \mathbf{0}$.

Clearly, the vector $(-1)\mathbf{a}$ is the opposite vector of $\mathbf{a}$, i.e. $(-1)\mathbf{a} = -\mathbf{a}$.

If $|\,\mathbf{a}\,| \neq 0$, then the vector

$$\mathbf{a}^0 = \frac{1}{|\,\mathbf{a}\,|}\,\mathbf{a} = \frac{\mathbf{a}}{|\,\mathbf{a}\,|}$$

is called the *unit vector* of $\mathbf{a}$ since its direction and sense are the same as those of $\mathbf{a}$ and

$$|\,\mathbf{a}^0\,| = \left|\frac{1}{|\,\mathbf{a}\,|}\,\mathbf{a}\right| = \frac{1}{|\,\mathbf{a}\,|}\,|\,\mathbf{a}\,| = 1\,.$$

It is clear from Definition **1.3** that the following theorem holds:

THEOREM 1.1. $\lambda\mathbf{a} = \mathbf{0}$ if and only if at least one of $\lambda = 0$ and $\mathbf{a} = \mathbf{0}$ holds.

Multiplication of vectors by scalars satisfies the following laws (λ μ and $\mathbf{a}$, $\mathbf{b}$ are arbitrary scalars and vectors, respectively):

$$\lambda\mathbf{a} = \mathbf{a}\lambda, \qquad \text{(commutativity)}, \qquad (1.9)$$

$$(\lambda\mu)\mathbf{a} = \lambda(\mu\mathbf{a}) = \lambda\mu\mathbf{a}, \qquad \text{(associativity)}, \qquad (1.10)$$

$$(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}, \qquad \text{(distributivity)}, \qquad (1.11)$$

$$\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}, \qquad \text{(distributivity)}. \qquad (1.12)$$

(1.9) is a convention. (1.10) is an obvious consequence of Definition 1.3. The distributive laws (1.11) and (1.12) ought to be proved, but instead of a formal proof see Figs 1.4 and 1.5, respectively.

THEOREM 1.2. If $\mathbf{a} \neq \mathbf{0}$ and $\lambda\mathbf{a} = \mu\mathbf{a}$, then $\lambda = \mu$.

Proof. From the condition given it follows that $\lambda\mathbf{a} - \mu\mathbf{a} = \mathbf{0}$, i.e. by (1.11) $(\lambda - \mu)\,\mathbf{a} = \mathbf{0}$, where $\mathbf{a} \neq \mathbf{0}$. Thus our statement is a consequence of Theorem **1.1.**!

THEOREM 1.3. The vector $\mathbf{a}$ is parallel to the vector $\mathbf{b}$ if and only if one of them is a scalar multiple of the other.

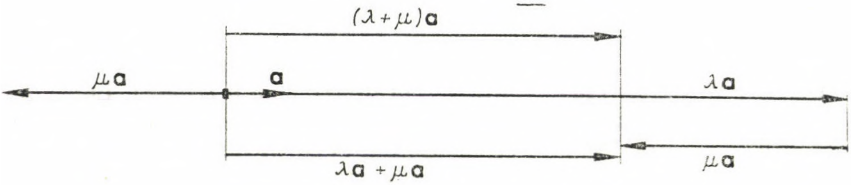

$(\lambda+\mu)\mathbf{a}$
$\mu\mathbf{a}$
$\mathbf{a}$
$\lambda\mathbf{a}$
$\lambda\mathbf{a}+\mu\mathbf{a}$
$\mu\mathbf{a}$

Fig. 1.4

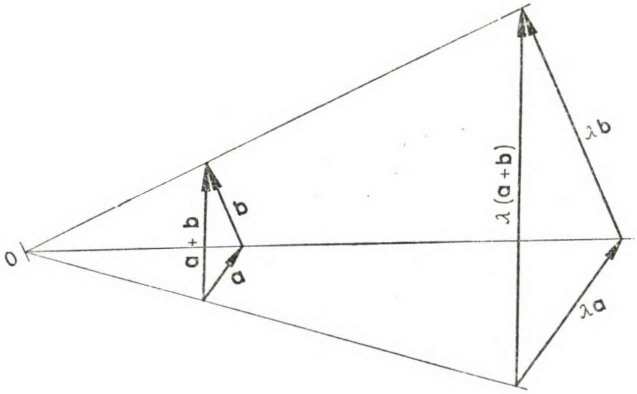

O
$\mathbf{a}+\mathbf{b}$
$\mathbf{b}$
$\mathbf{a}$
$\lambda(\mathbf{a}+\mathbf{b})$
$\lambda\mathbf{b}$
$\lambda\mathbf{a}$

Fig. 1.5

Proof. The condition is necessary. Let $\mathbf{a} \parallel \mathbf{b}$ and for instance $\mathbf{b} \neq \mathbf{0}$. Then, according to the two possible cases of their respective senses, either $\mathbf{a}^0 = \mathbf{b}^0$ or $\mathbf{a}^0 = -\mathbf{b}^0$, where $\mathbf{a}^0$ and $\mathbf{b}^0$ are the unit vectors of $\mathbf{a}$ and $\mathbf{b}$, respectively. (If $\mathbf{a}$ is the zero vector, then $\mathbf{a}^0$ is considered to be defined by either of these two equations.) In the first case,

$$\mathbf{a} = \mathbf{b}^0 \mid \mathbf{a} \mid = \frac{\mathbf{b}}{\mid \mathbf{b} \mid} \mid \mathbf{a} \mid = \frac{\mid \mathbf{a} \mid}{\mid \mathbf{b} \mid} \mathbf{b} .$$

In the second case,

$$\mathbf{a} = -\mathbf{b}^0 \mid \mathbf{a} \mid = -\frac{\mid \mathbf{a} \mid}{\mid \mathbf{b} \mid} \mathbf{b} .$$

If, finally, $\mathbf{b} = \mathbf{0}$, then $\mathbf{b} = 0\mathbf{a}$. !

For arbitrary vectors $\mathbf{a}$, $\mathbf{b}$ and scalars λ_1, λ_2 the vector $\lambda_1\mathbf{a} + \lambda_2\mathbf{b}$ is, clearly, coplanar with $\mathbf{a}$ and $\mathbf{b}$, since if $\mathbf{a} = \overrightarrow{PA}$, $\mathbf{b} = \overrightarrow{PB}$ and $\lambda_1\mathbf{a} + \lambda_2\mathbf{b} = \overrightarrow{PC}$, then the points P, A, B, C are in the same plane. The converse of this statement is the essence of

THEOREM 1.4. If the vectors **a** and **b** are not parallel and **v** is an arbitrary vector coplanar with **a** and **b**, then there exists a unique ordered pair of numbers (λ_1, λ_2) such that

$$\mathbf{v} = \lambda_1\mathbf{a} + \lambda_2\mathbf{b}, \tag{1.13}$$

or, in other words, **v** is a uniquely determined 'linear combination' of **a** and **b**.

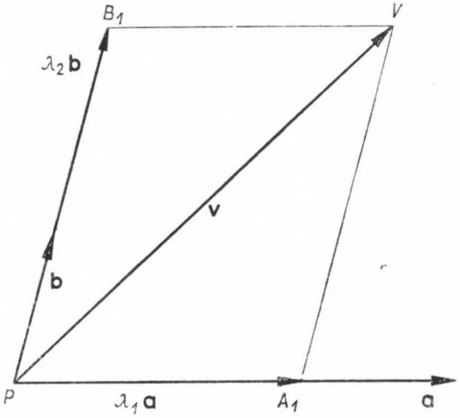


Fig. 1.6

Proof. In proving the existence of λ_1 and λ_2 satisfying (1.13), geometrical considerations will be used. Let $\mathbf{a} = \overrightarrow{PA}$, $\mathbf{b} = \overrightarrow{PB}$, and $\mathbf{v} = \overrightarrow{PV}$. Since neither **a** nor **b** is a zero vector (they are not parallel) the points P, A and P, B, respectively, define a straight line. Draw two straight lines through the point V parallel to PA and PB, respectively. The line through V parallel to PA intersects the line PB in a point B_1; the line through V parallel to PB intersects the line PA in a point A_1. By the parallelogram rule $\mathbf{v} = \overrightarrow{PA_1} + \overrightarrow{PB_1}$ (see Fig. 1.6) and by Theorem 1.3, $\overrightarrow{PA_1} = \lambda_1\mathbf{a}$ and $\overrightarrow{PB_1} = \lambda_2\mathbf{b}$. Thus, (1.13) holds.

We show indirectly that the ordered pair (λ_1, λ_2) in (1.13) is defined uniquely. Assume that, besides (1.13), we have $\mathbf{v} = \mu_1\mathbf{a} + \mu_2\mathbf{b}$. Then

$$\lambda_1\mathbf{a} + \lambda_2\mathbf{b} = \mu_1\mathbf{a} + \mu_2\mathbf{b},$$

or

$$(\lambda_1 - \mu_1)\mathbf{a} = (\mu_2 - \lambda_2)\mathbf{b}.$$

If $\lambda_1 \neq \mu_1$, then

$$\mathbf{a} = \frac{\mu_2 - \lambda_2}{\lambda_1 - \mu_1}\mathbf{b}$$

would hold. This would mean that $\mathbf{a} \parallel \mathbf{b}$, which contradicts the basic assumption on **a** and **b**. We get a similar contradiction if $\lambda_2 \neq \mu_2$ is assumed. Thus, $\mu_1 = \lambda_1$, $\mu_2 = \lambda_2$.!

In the previous theorem we considered coplanar vectors, i.e. vectors parallel to the same plane. The result is often expressed by saying that two non-parallel vectors *span their plane*. An analogous theorem is valid for vectors in space, namely

THEOREM 1.5. If the vectors **a**, **b**, and **c** are not parallel to the same plane and **v** is an arbitrary vector, then there exists a unique ordered triple of numbers $(\lambda_1, \lambda_2, \lambda_3)$ such that

$$\mathbf{v} = \lambda_1\mathbf{a} + \lambda_2\mathbf{b} + \lambda_3\mathbf{c}, \tag{1.14}$$

or, in other words, **v** is a uniquely determined 'linear combination' of **a**, **b**, and **c**.

Proof. We prove first the existence of λ_1, λ_2, λ_3 satisfying (1.14) by making use of the previous theorem. Since **a**, **b**, and **c** are not coplanar, none of them is a zero vector and no two of them are parallel. Let $\mathbf{a} = \overrightarrow{PA}$, $\mathbf{b} = \overrightarrow{PB}$, and $\mathbf{v} = \overrightarrow{PV}$. The points P, A, and B are not in a straight line, so they determine a plane. Draw a straight line through the point V parallel to the vector **c**. Since the latter vector is not parallel to the PAB plane, this line intersects the PAB plane in a point Q. Clearly, $\mathbf{v} = \overrightarrow{PQ} + \overrightarrow{QV}$. The vector $\overrightarrow{PQ}$ is coplanar with **a** and **b**, and so by the previous theorem $\overrightarrow{PQ} = \lambda_1\mathbf{a} + \lambda_2\mathbf{b}$. On the other hand, the vector $\overrightarrow{QV}$ is parallel to $\mathbf{c} \neq \mathbf{0}$, so by Theorem 1.3. $\overrightarrow{QV} = \lambda_3\mathbf{c}$. Hence (1.14) holds (Fig. 1.7).

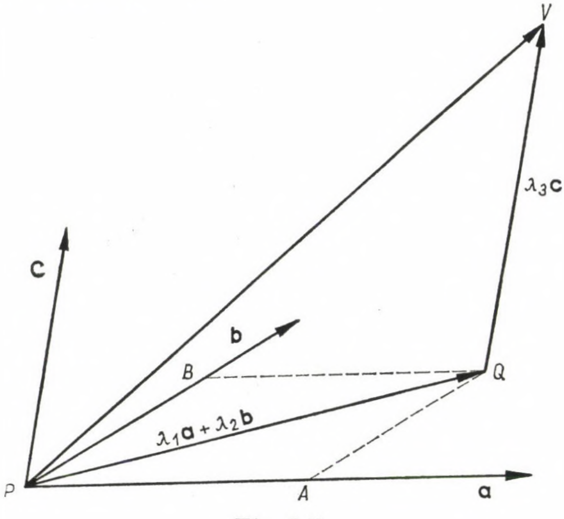


Fig. 1.7

To prove the uniqueness of the linear combination (1.14) assume that, besides (1.14), we have $\mathbf{v} = \mu_1\mathbf{a} + \mu_2\mathbf{b} + \mu_3\mathbf{c}$. Then

$$\lambda_1\mathbf{a} + \lambda_2\mathbf{b} + \lambda_3\mathbf{c} = \mu_1\mathbf{a} + \mu_2\mathbf{b} + \mu_3\mathbf{c},$$

or

$$(\lambda_1 - \mu_1)\mathbf{a} + (\lambda_2 - \mu_2)\mathbf{b} + (\lambda_3 - \mu_3)\mathbf{c} = \mathbf{0}.$$

If, for instance, $\lambda_1 \neq \mu_1$, then

$$\mathbf{a} = \frac{\mu_2 - \lambda_2}{\lambda_1 - \mu_1}\mathbf{b} + \frac{\mu_3 - \lambda_3}{\lambda_1 - \mu_1}\mathbf{c}$$

would hold. This would mean that **a**, **b**, and **c** are coplanar, and this is a contradiction. The assumptions $\lambda_2 \neq \mu_2$ or $\lambda_3 \neq \mu_3$ lead to similar contradictions. Thus, $\mu_1 = \lambda_1$, $\mu_2 = \lambda_2$, $\mu_3 = \lambda_3$.!

We express the fact contained in the previous theorem by saying that three non-coplanar vectors *span the space*.

Two (or more) parallel vectors, or three (or more) coplanar vectors, or four arbitrary vectors are said to be 'linearly dependent' since by Theorem 1.3, Theorem 1.4, and Theorem 1.5, respectively, at least one of them can be expressed as a linear combination of the rest. Two non-parallel vectors, or three non-coplanar vectors are said to be 'linearly independent' since none of them can be expressed as a linear combination of the rest (cf. the remark after Definition 9.1, and Definitions 16.6 and 16.7).

2. Multiplication of Vectors

There are two kinds of multiplication for vectors in ordinary space. The result of the first one is a scalar and that of the second one a vectors. Each concept will be introduced by an example.

EXAMPLE 2.1. If a constant force given by the vector **f** is acting on a point-like body and this body undergoes a displacement represented by the vector **r**, then the work L is the product of the magnitude of the force and the distance covered in the direction of the force. In other words, the work is equal to the magnitude of the displacement multiplied by the magnitude of the projection of the force in the direction of the displacement. If the angle between the force vector and the displacement vector is φ, then

$$L = |\mathbf{f}|\ |\mathbf{r}| \cos\varphi$$

(see Fig. 2.1; the work is positive if it is due to the force **f** and negative if it is done against this force).

Our first definition of vector multiplication yields a mathematical model expressing the work in terms of the force and the displacement (see the previous example).

DEFINITION 2.1. Let two arbitrary vectors and their angle be denoted by $\mathbf{a}$, $\mathbf{b}$, and φ, respectively; the number

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}| \cos \varphi \tag{2.1}$$

is called the *scalar product* (or *dot product*) of the vectors $\mathbf{a}$ and $\mathbf{b}$.

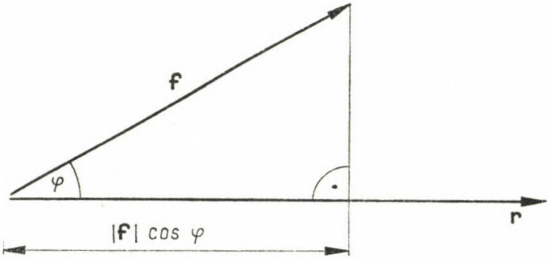


Fig. 2.1

The dot is often omitted and then $\mathbf{ab}$ has the same meaning as $\mathbf{a} \cdot \mathbf{b}$. If in the previous definition one of the vectors is the zero vector, the angle between the vectors is not defined. However, in this case one of the lengths on the right-hand side of (2.1) is zero and the scalar product becomes zero.

In discussing the operational laws of the scalar product we shall need the following:

LEMMA 2.1. For any two vectors $\mathbf{a}$ and $\mathbf{b}$ either of them, say $\mathbf{a}$, can be expressed as the sum of a vector parallel to the other $\mathbf{b}$ and a vector orthogonal to $\mathbf{b}$; if $\mathbf{b} \neq \mathbf{0}$, then in this decomposition of $\mathbf{a}$ the component of $\mathbf{a}$ parallel to $\mathbf{b}$ is

$$\mathbf{a}_p = (\mathbf{a} \cdot \mathbf{b}^0)\, \mathbf{b}^0. \tag{2.2}$$

Proof. If $\mathbf{b} = \mathbf{0}$, then the 'trivial decomposition' $\mathbf{a} = \mathbf{0} + \mathbf{a}$ is the one which we are looking for. Let $\mathbf{b} \neq \mathbf{0}$ and $\mathbf{a} = \overrightarrow{PA}$, $\mathbf{b} = \overrightarrow{PB}$. The projection of the point A on the line determined by $\overrightarrow{PB}$ will be denoted by C (see Fig. 2.2). The decomposition needed is $\mathbf{a} = \overrightarrow{PC} + \overrightarrow{CA}$. If the angle of $\mathbf{a}$ and $\mathbf{b}$ is denoted by φ, then

$$|\overrightarrow{PC}| = ||\mathbf{a}| \cos \varphi| = ||\mathbf{a}|\ |\mathbf{b}^0| \cos \varphi| = |\mathbf{a} \cdot \mathbf{b}^0|.$$

The vector $\overrightarrow{PC}$ is parallel to $\mathbf{b}$ and of the same or opposite sense, depending on whether φ is an acute or an obtuse angle ($\cos \varphi > 0$, or $\cos \varphi <$

0, respectively). If $\varphi = \pi/2$, i.e. $\cos\varphi = 0$, then again the trivial decomposition is the one needed. If φ is an acute angle, then $\overrightarrow{PC} = |\mathbf{a}\cdot\mathbf{b}^0|\,\mathbf{b}^0 = (\mathbf{a}\cdot\mathbf{b}^0)\,\mathbf{b}^0$. If φ is an obtuse angle, then $\overrightarrow{PC} = |\mathbf{a}\cdot\mathbf{b}^0|\,(-\mathbf{b}^0) = -(\mathbf{a}\cdot\mathbf{b}^0)\,(-\mathbf{b}^0) = (\mathbf{a}\cdot\mathbf{b}^0)\mathbf{b}^0$. !

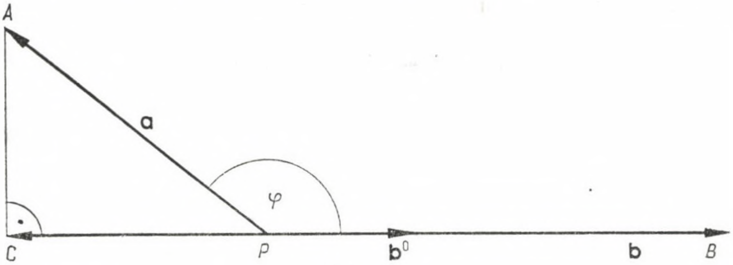


Fig. 2.2

The scalar multiplication of vectors satisfies the following laws ($\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and λ are arbitrary vectors and a scalar, respectively):

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}, \qquad \text{(commutativity)}, \tag{2.3}$$

$$(\lambda\mathbf{a})\cdot\mathbf{b} = \lambda(\mathbf{a}\cdot\mathbf{b}), \tag{2.4}$$

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}, \qquad \text{(distributivity)}. \tag{2.5}$$

Identity (2.3) is an immediate consequence of Definition 2.1.

Proof of (2.4). $\cos(\mathbf{a};\mathbf{b})$ etc. means the cosine of the angle of the two vectors. If $\lambda = 0$, then (2.4) is obvious. If $\lambda \neq 0$, then $(\lambda\mathbf{a})\cdot\mathbf{b} = |\lambda\mathbf{a}|\,|\mathbf{b}|\cos(\lambda\mathbf{a};\mathbf{b}) = |\lambda|\,|\mathbf{a}|\,|\mathbf{b}|\cos(\lambda\mathbf{a};\mathbf{b})$. If $\lambda > 0$, then $|\lambda| = \lambda$ and $\cos(\lambda\mathbf{a};\mathbf{b}) = \cos(\mathbf{a};\mathbf{b})$; thus $|\lambda|\,|\mathbf{a}|\,|\mathbf{b}|\cos(\lambda\mathbf{a};\mathbf{b}) = \lambda(\mathbf{a}\cdot\mathbf{b})$. If $\lambda < 0$, then $|\lambda| = -\lambda$ and $\cos(\lambda\mathbf{a};\mathbf{b}) = -\cos(\mathbf{a};\mathbf{b})$; thus $|\lambda|\,|\mathbf{a}|\,|\mathbf{b}|\cos(\lambda\mathbf{a};\mathbf{b}) = -\lambda\,|\mathbf{a}|\,|\mathbf{b}|\,(-\cos(\mathbf{a};\mathbf{b})) = \lambda(\mathbf{a}\cdot\mathbf{b})$.!

Proof of (2.5). If $\mathbf{a} = \mathbf{0}$, then (2.5) is obviously true, since on both sides we have zero. If $\mathbf{a} \neq \mathbf{0}$, consider its unit vector $\mathbf{a}^0$. We shall prove first the identity

$$\mathbf{a}^0\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}^0\cdot\mathbf{b} + \mathbf{a}^0\cdot\mathbf{c}. \tag{2.6}$$

Let $\mathbf{b} = \overrightarrow{AB}$, $\mathbf{c} = \overrightarrow{BC}$; then $\mathbf{b}+\mathbf{c} = \overrightarrow{AC}$. Project the triangle ABC to the straight line passing through the point A in the direction given by $\mathbf{a}^0$. The projections of the sides of this triangle represent the components parallel to $\mathbf{a}^0$ of the corresponding vectors. In other words if B' and C' are the projections of the points B and C, respectively, then $\mathbf{b}_p = \overrightarrow{AB'}$, $\mathbf{c}_p = \overrightarrow{B'C'}$ and $(\mathbf{b}+\mathbf{c})_p = \overrightarrow{AC'}$ (see Fig. 2.3). Thus

$$(\mathbf{b}+\mathbf{c})_p = \mathbf{b}_p + \mathbf{c}_p.$$

By Lemma 2.1 (making use of the commutative law (2.3)) this last identity can be written in the form

$$(\mathbf{a}^0 \cdot (\mathbf{b} + \mathbf{c}))\mathbf{a}^0 = (\mathbf{a}^0 \cdot \mathbf{b})\mathbf{a}^0 + (\mathbf{a}^0 \cdot \mathbf{c})\mathbf{a}^0.$$

Making use of the distributive law (1.9) we have

$$[\mathbf{a}^0 \cdot (\mathbf{b} + \mathbf{c}) - \mathbf{a}^0 \cdot \mathbf{b} - \mathbf{a}^0 \cdot \mathbf{c}]\mathbf{a}^0 = \mathbf{0}.$$

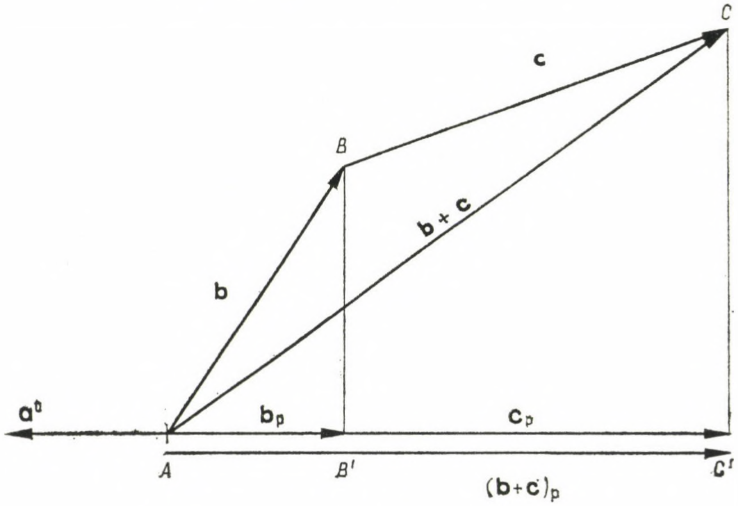


Fig. 2.3

Hence, by Theorem 1.1, (2.6) holds. Multiplying (2.6) by $|\mathbf{a}|$ and applying (2.4) to each term we get (2.5). !

It is easy to prove by mathematical induction that an identity similar to (2.5) holds if **a** is multiplied by the sum of any finite number of vectors.

The scalar product of **a** and itself is called the *square of* **a**. By Definition 2.1

$$(\mathbf{a})^2 = \mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 \cos 0 = |\mathbf{a}|^2, \tag{2.7}$$

hence

$$|\mathbf{a}| = \sqrt{(\mathbf{a})^2}. \tag{2.8}$$

It can be seen from Definition 2.1 that for any two vectors **a** and **b** we have the important inequality

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}|\ |\mathbf{b}|, \tag{2.9}$$

and that the sign of equality is valid in (2.9) if and only if the angle of **a** and **b** is 0 or π, i.e. **a** and **b** are parallel.

The following theorem is a simple consequence of the definition of the scalar product, but it is of basic importance and frequently used.

THEOREM 2.2. Two vectors are orthogonal if and only if their scalar product is zero.

Proof. Assume that $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\varphi = 0$, where φ is the angle of the two vectors. If one of the vectors is the zero vector then they are orthogonal, since the zero vector is orthogonal to every vector. If neither of the vectors is zero, then $\cos\varphi = 0$, i.e. $\varphi = \pi/2$. Conversely, if **a** is orthogonal to **b**, then one of them is zero, or their angle $\varphi = \pi/2$. In both cases $\mathbf{a} \cdot \mathbf{b} = 0$. !

COROLLARY 2.3. If $\mathbf{a}^1$, $\mathbf{a}^2$, and $\mathbf{a}^3$ are non-coplanar vectors and the vector **v** is orthogonal to all three, then $\mathbf{v} = \mathbf{0}$.

Proof. By Theorem 1.5 $\mathbf{v} = \lambda_1\mathbf{a}^1 + \lambda_2\mathbf{a}^2 + \lambda_3\mathbf{a}^3$. Take the scalar product of **v** with both sides of this equation. Then making use of (2.5), (2.4), and Theorem 2.2.

$$(\mathbf{v})^2 = \lambda_1\mathbf{a}^1 \cdot \mathbf{v} + \lambda_2\mathbf{a}^2 \cdot \mathbf{v} + \lambda_3\mathbf{a}^3 \cdot \mathbf{v} = 0. \;!$$

There is a second way (in addition to scalar multiplication) of defining the product of two vectors. The product in this case will be a vector. We start again with an example taken from mechanics.

EXAMPLE 2.2. The moment of a force about a point is defined, in general, as a vector. If a rigid body is fixed at a point O and a force given by the vector **f** is acting at a point P of the body, then the moment (about the point O) of the force **f** acting at P is represented by a vector **m**. The length of **m** (the magnitude of the moment) is $|\mathbf{f}| \cdot |\overrightarrow{OP}| \sin\varphi$, where φ is the angle of **f** and $\overrightarrow{OP}$. Since $|\overrightarrow{OP}| \sin\varphi$ is the distance of the point O from the line passing through P in the direction of **f** (see Fig. 2.4), the magnitude of the moment is the product of the magnitude and the arm of the force. The vector **m** is considered to be orthogonal to the vectors $\overrightarrow{OP}$ and **f**. The sense of the

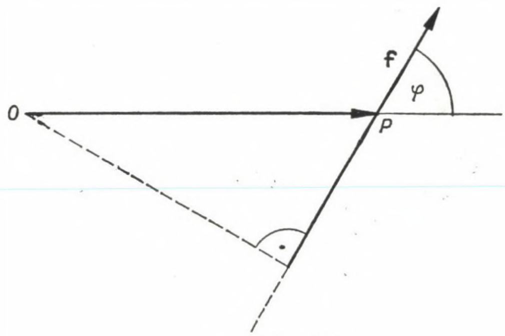


Fig. 2.4

moment is chosen so that, looking at the body in the direction opposite to this sense, the body is rotating (or trying to rotate) due to the force **f** about the point O in a 'positive sense', i.e. counter-clockwise. From this definition of the moment we find that if different forces are acting at different points on the rigid body, then the sum of the corresponding moment vectors yields a vector which represents the resultant moment. Then the resultant moment can be substituted for the system of moments.

DEFINITION 2.2. Let there be given an ordered triple of non-coplanar vectors **a**, **b**, **c**; if these vectors are drawn from the same point, that is to say, if $\mathbf{a} = \overrightarrow{PA}$, $\mathbf{b} = \overrightarrow{PB}$, $\mathbf{c} = \overrightarrow{PC}$, and looking in the direction opposite to the sense of **c** (i.e. from the point C) at **a** and **b** (i.e. at the plane determined by the points P, A, B) **a** can be rotated into the direction and sense of **b** by a counter-clockwise rotation of angle less than π, then we say that these vectors form a *right-handed* (or *dextral*) *system*.

The name given to a right-handed system of vectors **a**, **b**, **c** in the previous definition comes from the fact that the thumb, index, and middle fingers of the right hand can be extended so that they have the direction (and sense) of **a**, **b**, and **c**, respectively. It is easy to see that if **a**, **b**, **c** form a right-handed system, then **b**, **c**, **a** and **c**, **a**, **b** also form right-handed systems; **a**, **c**, **b**, etc., form a *left-handed* (or *sinistral*) *system*.

DEFINITION 2.3. Let **a** and **b** be arbitrary vectors and φ their angle; the *vectorial* (or *cross*) *product* of **a** and **b** (taken in a fixed order) is the vector denoted by $\mathbf{a}\times\mathbf{b}$ whose length is given by $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\varphi$, whose direction is orthogonal to both **a** and **b** and whose sense is such that (if $\mathbf{a}\times\mathbf{b}$ is non-zero) **a**, **b**, and $\mathbf{a}\times\mathbf{b}$ form a right-handed system.

If the vectors **a** and **b** are drawn from the same point, that is to say, $\mathbf{a} = \overrightarrow{PA}$ and $\mathbf{b} = \overrightarrow{PB}$, and their sum is $\mathbf{a} + \mathbf{b} = \overrightarrow{PC}$, then we say that the vectors **a** and **b** *span the parallelogram* with vertices P, A, B, C (see Fig. 2.5). Obviously, the length of the cross product $|\mathbf{a}\times\mathbf{b}|$ is equal to the area of this parallelogram.

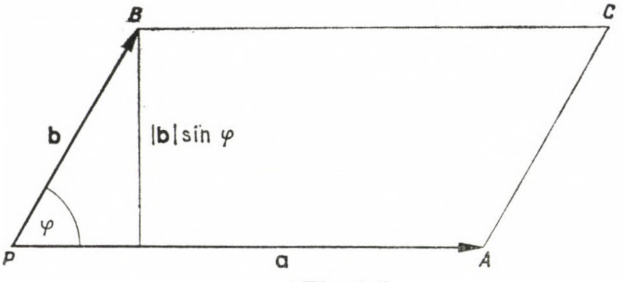


Fig. 2.5

We now discuss the operational laws concerning the cross product of vectors.

From Definition 2.3 the identity

$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a} \tag{2.10}$$

follows at once. This means that the cross product is not a commutative multiplication.

The following theorem is also a simple consequence of the definition (it can be proved similarly to the proof of Theorem 2.2).

THEOREM 2.4. Two vectors are parallel if and only if their cross product is zero.

For arbitrary vectors **a** and **b** and scalar number λ,

$$\lambda(\mathbf{a}\times\mathbf{b}) = (\lambda\mathbf{a})\times\mathbf{b} = \mathbf{a}\times(\lambda\mathbf{b}). \tag{2.11}$$

Proof of (2.11). If **a** and **b** are parallel or $\lambda = 0$, then (2.11) is obvious. If **a** is not parallel to **b**, then it is sufficient to prove (2.11) for $\lambda > 0$ and for $\lambda = -1$, since if $\lambda < 0$, then the multiplication can be performed in two steps, namely multiplying first by $|\lambda|$ and then by -1. If $\lambda > 0$, then the vectors $\lambda\mathbf{a}$ and $\lambda\mathbf{b}$ have the same sense as the vectors **a** and **b**, respectively. Thus (2.11) follows from Definitions 1.3 and 2.3 at once. If $\lambda = -1$ and α denotes the angle of **a** and **b**, then, because of the identity $\sin\alpha = \sin(\pi-\alpha)$, the length and direction of the vectors $(-\mathbf{a})\times\mathbf{b}$ and $\mathbf{a}\times(-\mathbf{b})$ are equal to the length and direction of $\mathbf{a}\times\mathbf{b}$. At the same time the sense of the first two vectors is opposite to the sense of the last, i.e.

$$(-\mathbf{a})\times\mathbf{b} = \mathbf{a}\times(-\mathbf{b}) = -(\mathbf{a}\times\mathbf{b}).\,!$$

If three vectors **a**, **b**, and **c** are given and we take first the cross product of the first two and then the scalar product of $\mathbf{a}\times\mathbf{b}$ with **c**, we get a scalar number. The number

$$(\mathbf{abc}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} \tag{2.12}$$

is called the *scalar triple product* of the vectors **a**, **b**, and **c**.

Let us measure the vectors **a**, **b**, and **c** from the same point P_0 and let

$$\mathbf{a} = \overrightarrow{P_0P_1},\ \mathbf{b} = \overrightarrow{P_0P_2},\ \mathbf{c} = \overrightarrow{P_0P_3}.$$

There is exactly one parallelepiped with edges P_0P_1, P_0P_2, P_0P_3 running into the vertex P_0. We say that the vectors **a**, **b**, and **c** span this parallelepiped. The absolute value of (2.12) is $|(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}| = |\mathbf{a}\times\mathbf{b}|\,|(\mathbf{a}\times\mathbf{b})^0\cdot\mathbf{c}|$ where $(\mathbf{a}\times\mathbf{b})^0$ is the unit vector of $\mathbf{a}\times\mathbf{b}$. (provided that it exists, i.e. $\mathbf{a}\times\mathbf{b}\neq 0$). Since the area of the parallelogram spanned by the vectors **a** and **b** is $|\mathbf{a}\times\mathbf{b}|$ and the height corresponding to this side is the projection of the vector **c** onto the direction of $\mathbf{a}\times\mathbf{b}$, i.e. $(\mathbf{a}\times\mathbf{b})^0\cdot\mathbf{c}$, the absolute value of the scalar triple product of **a**, **b**, and **c** is equal to the volume of the parallelepiped spanned by these vectors. An important consequence of these geometrical considerations is the following:

THEOREM 2.5. Three vectors are coplanar if and only if their scalar triple product is zero.

Proof. If **a**, **b**, and **c** are coplanar vectors then the parallelepiped spanned by them is degenerate–its volume is zero–thus $(\mathbf{abc}) = 0$. Conversely, if $(\mathbf{abc}) = 0$ and $\mathbf{a}\times\mathbf{b} = 0$ then by Theorem 2.4 **a** and **b** are parallel, thus **a**, **b**, and **c** are coplanar; if $\mathbf{a}\times\mathbf{b} \neq 0$ then by Theorem 2.2 **c** is orthogonal to $\mathbf{a}\times\mathbf{b}$, i.e. **c** is parallel to the plane of the parallelogram spanned by **a** and **b**, thus **a**, **b**, and **c** are coplanar. !

The notation introduced in (2.12) suggests that the 'dot' and the 'cross' can be interchanged in a scalar triple product, that is to say we have the identity

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}). \tag{2.13}$$

In fact, $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| = |\mathbf{b}\times\mathbf{c}|\,|(\mathbf{b}\times\mathbf{c})^0\cdot\mathbf{a}|$. Since $|\mathbf{b}\times\mathbf{c}|$ is the area of a side of the parallelepiped spanned by the three vectors and $|(\mathbf{b}\times\mathbf{c})^0\cdot\mathbf{a}|$ is the height corresponding to this side, the absolute value of the right-hand side of (2.13) is equal to the volume of the parallelepiped, i.e. it is equal to the absolute value of the left-hand side. If **a**, **b**, and **c** form a dextral system, then $\mathbf{a}\times\mathbf{b}$ and **c** contain an acute angle and $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ is therefore positive; but in this case also **b**, **c**, and **a** form a dextral system, thus $(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}$ is also positive. One can see similarly that if the left-hand side of (2.13) is negative then so also is the right-hand side.

Besides (2.13) we have the identity

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = (\mathbf{c}\times\mathbf{a})\cdot\mathbf{b} \tag{2.14}$$

which can be proved in the same way. Identities (2.13) and (2.14) are summed up by saying that the scalar triple product remains unaltered if the terms undergo a 'cyclic permutation'

$$\begin{tikzcd} \mathbf{a} \arrow[rr] & & \mathbf{b} \arrow[dl] \\ & \mathbf{c} \arrow[ul] & \end{tikzcd}$$

:

$$(\mathbf{abc}) = (\mathbf{bca}) = (\mathbf{cab}). \tag{2.15}$$

On the other hand, if the cycle of the terms is reversed then the scalar triple product changes sign:

$$(\mathbf{cba}) = (\mathbf{bac}) = (\mathbf{acb}) = -(\mathbf{abc}). \tag{2.16}$$

Now we can prove the *distributive law* for the cross product of vectors. We have the distributive identity

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}. \tag{2.17}$$

Proof of (2.17). It will be shown that the vector

$$\mathbf{v}=\mathbf{a}\times(\mathbf{b}+\mathbf{c})-\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}$$

is the zero vector. This, obviously, proves (2.17). Consider an arbitrary vector $\mathbf{u}$ and evaluate the following dot product making use of the distributive law (2.5) and of (2.14):

$$\begin{aligned}\mathbf{v}\cdot\mathbf{u}&=[\mathbf{a}\times(\mathbf{b}+\mathbf{c})]\cdot\mathbf{u}-(\mathbf{a}\times\mathbf{b})\cdot\mathbf{u}-(\mathbf{a}\times\mathbf{c})\cdot\mathbf{u}\\&=(\mathbf{u}\times\mathbf{a})\cdot(\mathbf{b}+\mathbf{c})-(\mathbf{u}\times\mathbf{a})\cdot\mathbf{b}-(\mathbf{u}\times\mathbf{a})\cdot\mathbf{c}=0.\end{aligned}$$

Thus $\mathbf{v}$ is orthogonal to every vector and by Corollary 2.3 this means that $\mathbf{v}=\mathbf{0}$. !

Multiplying (2.17) by -1 and reversing the order of factors in all the cross products there, we get

$$(\mathbf{b}+\mathbf{c})\times\mathbf{a}=\mathbf{b}\times\mathbf{a}+\mathbf{c}\times\mathbf{a}. \tag{2.18}$$

It is easy to see that no associative law holds for the cross product. Instead the following identities hold (for vectorial triple products):

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\,\mathbf{a} \tag{2.19}$$

and

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}. \tag{2.20}$$

In proving (2.19) and (2.20) we shall make use of the following:

LEMMA 2.6. Assume that $\mathbf{a}$ and $\mathbf{b}$ are arbitrary non-parallel vectors; then $\mathbf{a}$ can be expressed as the sum of a vector parallel to $\mathbf{b}$ and a vector orthogonal to $\mathbf{b}$; in this decomposition of $\mathbf{a}$ the component of $\mathbf{a}$ orthogonal to $\mathbf{b}$ is

$$\mathbf{a}_{\mathrm{or}}=(\mathbf{b}^0\times\mathbf{a})\times\mathbf{b}^0, \tag{2.21}$$

(cf. Lemma 2.1).

Proof. By Lemma 2.1 $\mathbf{a}_{\mathrm{or}}=\mathbf{a}-\mathbf{a}_p=\mathbf{a}-(\mathbf{a}\cdot\mathbf{b}^0)\mathbf{b}^0$ (see Fig. 2.2). Denote by φ the angle of $\mathbf{a}$ and $\mathbf{b}$ (our assumption implies that neither $\mathbf{a}$ nor $\mathbf{b}$ is a zero vector), then $|\mathbf{a}_{\mathrm{or}}|=|\mathbf{a}|\sin\varphi$. On the other hand, since $\mathbf{b}^0\times\mathbf{a}$ is orthogonal to $\mathbf{b}^0$, therefore $|(\mathbf{b}^0\times\mathbf{a})\times\mathbf{b}^0|=|\mathbf{b}^0\times\mathbf{a}|\,|\mathbf{b}^0|\sin\pi/2=|\mathbf{b}^0\times\mathbf{a}|=|\mathbf{a}|\sin\varphi$. Thus $|\mathbf{a}_{\mathrm{or}}|=|(\mathbf{b}^0\times\mathbf{a})\times\mathbf{b}^0|$. The direction of these vectors is also the same since both are in the plane spanned by $\mathbf{a}$ and $\mathbf{b}^0$ (the latter is in this plane because it is orthogonal to $\mathbf{b}^0\times\mathbf{a}$) and both are orthogonal to $\mathbf{b}^0$. The sense of these vectors is either the same or opposite. We are going to show that their angle is less than π, thus it is zero. The triples $\mathbf{b}^0 \times \mathbf{a}$, $\mathbf{b}^0$, $\mathbf{a}$ and $\mathbf{b}^0 \times \mathbf{a}$, $\mathbf{b}^0$, $(\mathbf{b}^0 \times \mathbf{a}) \times \mathbf{b}^0$ both form a right-handed system. The first and second vectors in both triples are the same and $(\mathbf{b}^0 \times \mathbf{a}) \times \mathbf{b}^0$ is orthogonal to the plane spanned by $\mathbf{b}^0 \times \mathbf{a}$ and $\mathbf{b}^0$, thus the angle of $(\mathbf{b}^0 \times \mathbf{a}) \times \mathbf{b}^0$ and $\mathbf{a}$ is less than $\pi/2$. On the other hand, the angle of $\mathbf{a}_{\mathrm{or}}$ and $\mathbf{a}$ is also less than $\pi/2$, since this is an angle of the right-angled vector triangle representing the sum $\mathbf{a} = \mathbf{a}_{\mathrm{or}} + \mathbf{a}_p$. Thus the angle of $(\mathbf{b}^0 \times \mathbf{a}) \times \mathbf{b}^0$ and $\mathbf{a}_{\mathrm{or}}$ is less than $\pi/2 + \pi/2 = \pi$, and therefore it is zero. Since the length, the direction, and the sense of these vectors is the same, (2.21) holds. !

Proof of (2.19). If one of the vectors is zero then both sides of (2.19) are zero and (2.19) holds. Assume next that neither of the vectors is zero but $\mathbf{a}$ is parallel to $\mathbf{b}$. Then the left-hand side of (2.19) is zero. Substituting $\mathbf{a} = |\mathbf{a}|\, \mathbf{a}^0$ and $\mathbf{b} = \pm\, |\mathbf{b}|\, \mathbf{a}^0$ into the right-hand side of (2.19) we clearly get zero again. Assume finally that neither of the vectors is zero and $\mathbf{a}$ is not parallel to $\mathbf{b}$. It will be shown that the vector

$$\mathbf{v} = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} + (\mathbf{b} \cdot \mathbf{c})\, \mathbf{a}$$

is the zero vector. This obviously proves (2.19). According to the last assumption the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a} \times \mathbf{b}$ span the space. We show that $\mathbf{v}$ is orthogonal to all three:

$$\mathbf{v} \cdot \mathbf{a} = \mathbf{v} \cdot \mathbf{b} = \mathbf{v} \cdot (\mathbf{a} \times \mathbf{b}) = 0.$$

Making use of (2.15) and the operational laws of the scalar and vectorial product

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{a} &= \big((\mathbf{a} \times \mathbf{b})\, \mathbf{c}\mathbf{a}\big) - (\mathbf{a} \cdot \mathbf{c})\ (\mathbf{b} \cdot \mathbf{a}) + (\mathbf{b} \cdot \mathbf{c})\ (\mathbf{a} \cdot \mathbf{a}) \\
&= \big(\mathbf{a} \times (\mathbf{a} \times \mathbf{b})\big) \cdot \mathbf{c} - (\mathbf{b} \cdot \mathbf{a})\ (\mathbf{a} \cdot \mathbf{c}) + \mathbf{a}^2 (\mathbf{b} \cdot \mathbf{c}) \\
&= |\mathbf{a}|^2\, [\mathbf{a}^0 \times (\mathbf{a}^0 \times \mathbf{b}) - (\mathbf{b} \cdot \mathbf{a}^0)\, \mathbf{a}^0 + \mathbf{b}] \cdot \mathbf{c} \\
&= |\mathbf{a}|^2\, \{\mathbf{b} - [(\mathbf{a}^0 \cdot \mathbf{b})\, \mathbf{a}^0 + (\mathbf{a}^0 \times \mathbf{b}) \times \mathbf{a}^0]\} \cdot \mathbf{c}.
\end{aligned}$$

Applying Lemmas 2.1 and 2.6 while interchanging the role of **a** and **b** in these lemmas we have $\mathbf{v} \cdot \mathbf{a} = |\mathbf{a}|^2 \{\mathbf{b} - \mathbf{b}\} \cdot \mathbf{c} = 0$.

Similarly, we also obtain

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{b} &= \big((\mathbf{a} \times \mathbf{b})\, \mathbf{c}\mathbf{b}\big) - (\mathbf{a} \cdot \mathbf{c})\ (\mathbf{b} \cdot \mathbf{b}) + (\mathbf{b} \cdot \mathbf{c})\ (\mathbf{a} \cdot \mathbf{b}) \\
&= |\mathbf{b}|^2\, [\mathbf{b}^0 \times (\mathbf{a} \times \mathbf{b}^0) - \mathbf{a} + (\mathbf{a} \cdot \mathbf{b}^0)\, \mathbf{b}^0] \cdot \mathbf{c} \\
&= |\mathbf{b}|^2\, [(\mathbf{a} \cdot \mathbf{b}^0)\, \mathbf{b}^0 + (\mathbf{b}^0 \times \mathbf{a}) \times \mathbf{b}^0 - \mathbf{a}] \cdot \mathbf{c} \\
&= |\mathbf{b}|^2\, [\mathbf{a} - \mathbf{a}] \cdot \mathbf{c} = 0.
\end{aligned}$$

Lastly, by Theorem 2.5

$$\mathbf{v} \cdot (\mathbf{a}\times\mathbf{b}) = \big((\mathbf{a}\times\mathbf{b})\,\mathbf{c}(\mathbf{a}\times\mathbf{b})\big) - (\mathbf{a} \cdot \mathbf{c})\ (\mathbf{bab}) + (\mathbf{b} \cdot \mathbf{c})\ (\mathbf{aab}) = 0,$$

since in all the scalar triple products the vectors are coplanar.

Since $\mathbf{v}$ is orthogonal to three non-coplanar vectors, our statement follows from Corollary 2.3. !

Proof of (2.20). Interchanging the factors on the left-hand side of (2.20)

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = -(\mathbf{b}\times\mathbf{c})\times\mathbf{a}.$$

If we write **b**, **c**, and **a** for **a**, **b**, and **c**, respectively, in (2.19), we get

$$-(\mathbf{b}\times\mathbf{c})\times\mathbf{a} = -[(\mathbf{b} \cdot \mathbf{a})\,\mathbf{c} - (\mathbf{c} \cdot \mathbf{a})\,\mathbf{b}] = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$$

and this amounts to (2.20) if the previous identity is taken into account. !

Note that division has not, and cannot, be defined for vectors. Division of (real) numbers is based on the fact that if $a \neq 0$ and b are given numbers, then the equation $ax = b$ has exactly one solution, $x = b/a$. In particular if $b = 0$, then from $ax = 0$ it follows that $x = 0$. Now this is not the case with vectors. From $\mathbf{a} \cdot \mathbf{x} = 0$, or from $\mathbf{a}\times\mathbf{x} = \mathbf{0}$, the result $\mathbf{x} = \mathbf{0}$ does not follow since the first equation is satisfied by all vectors orthogonal to **a** and the second by all vectors parallel to **a**.

3. Bases. Coordinates

The vector operations defined in the previous paragraphs cannot easily be performed using only the definitions themselves and the operational laws. To represent a vector by its length, direction, and sense also seems awkward. The use of vectors for practical purposes is made possible by the fact that, from Theorem 1.5, there is a one-to-one correspondence between the set of vectors and the set of ordered triples of numbers.

Definition 3.1. If the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are not coplanar then we say that they form a *base* in the space; $\mathbf{e}_k$, $k = 1, 2, 3$ are called the vectors of the base, or *base vectors*.

If the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ form a base then by Theorem 1.5 any vector $\mathbf{v}$ has exactly one decomposition of the form

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3. \tag{3.1}$$

If the ordered triple of numbers (v_1, v_2, v_3) is made to correspond to the vector $\mathbf{v}$ then, in the fixed base, this is a one-to-one correspondence between the set of vectors and the set of ordered triples. We call v_1, v_2, and v_3 the *coordinates* of the vector $\mathbf{v}$ and denote the correspondence between $\mathbf{v}$ and its coordinate triple by writing $\mathbf{v} \simeq (v_1, v_2, v_3)$.

Obviously, two vectors given with their coordinates in the same base are equal if and only if their respective coordinates are equal.

Definition 3.2. If the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ form a base and they are orthogonal to each other then we say that their base is *orthogonal*; if moreover $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are unit vectors then their base is called *orthonormal.*

In the rest of this chapter we assume that $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ form an orthonormal base and at the same time a right-handed system. A right-handed orthonormal base is characterized by the following conditions:

$$(\mathbf{e}_1)^2 = (\mathbf{e}_2)^2 = (\mathbf{e}_3)^2 = 1 \,, \tag{3.2}$$

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_3 = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0 \,, \tag{3.3}$$

$$\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2 \,. \tag{3.4}$$

Obviously the conditions

$$\mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_3, \; \mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_1 \tag{3.5}$$

and

$$\mathbf{e}_1 \times \mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_3 = \mathbf{0} \tag{3.6}$$

also hold.

If the angle of the non-zero vector $\mathbf{v}$ and $\mathbf{e}_k$ is denoted by α_k $(k = 1, 2, 3)$ and the scalar product is formed of $\mathbf{e}_k$ with both sides of (3.1), we get

$$v_k = |\, \mathbf{v} \,| \cos \alpha_k, \; k = 1, 2, 3, \tag{3.7}$$

or

$$\cos \alpha_1 = \frac{v_1}{|\, \mathbf{v} \,|}, \quad \cos \alpha_2 = \frac{v_2}{|\, \mathbf{v} \,|}, \quad \cos \alpha_3 = \frac{v_3}{|\, \mathbf{v} \,|} \,.$$

These are the coordinates of the unit vector of $\mathbf{v}$:

$$\mathbf{v}^0 = \frac{\mathbf{v}}{|\, \mathbf{v} \,|} \simeq \left(\frac{v_1}{|\, \mathbf{v} \,|}, \frac{v_2}{|\, \mathbf{v} \,|}, \frac{v_3}{|\, \mathbf{v} \,|} \right) = (\cos \alpha_1, \; \cos \alpha_2, \; \cos \alpha_3) \,. \tag{3.8}$$

Hence the coordinates of a unit vector are equal to the cosines of its respective angles formed with the vectors of the base. That is why the coordinates of the unit vector $\mathbf{v}^0$ are sometimes called the *direction cosines* of the vector $\mathbf{v}$.

We now show how to perform the vector operations if the vectors are given with their coordinates in a right-handed orthonormal base. Let

$$\mathbf{a} \simeq (a_1, a_2, a_3), \; \mathbf{b} \simeq (b_1, b_2, b_3), \; \mathbf{c} \simeq (c_1, c_2, c_3)$$

be arbitrary vectors and λ an arbitrary scalar number. Then applying the operational laws

$$\begin{aligned}\mathbf{a} \pm \mathbf{b} &= (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) \pm (b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3)\\ &= (a_1 \pm b_1)\,\mathbf{e}_1 + (a_2 \pm b_2)\,\mathbf{e}_2 + (a_3 \pm b_3)\,\mathbf{e}_3,\end{aligned}$$

that is,

$$\mathbf{a} \pm \mathbf{b} \simeq (a_1 \pm b_1,\ a_2 \pm b_2,\ a_3 \pm b_3). \tag{3.9}$$

$$\begin{aligned}\lambda\mathbf{a} &= \lambda(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3)\\ &= \lambda a_1\mathbf{e}_1 + \lambda a_2\mathbf{e}_2 + \lambda a_3\mathbf{e}_3,\end{aligned}$$

that is,

$$\lambda\mathbf{a} \simeq (\lambda a_1, \lambda a_2, \lambda a_3). \tag{3.10}$$

The scalar product of **a** and **b** is obtained by applying the properties (2.3)–(2.5) of the scalar product and the properties (3.2)–(3.3) of the base:

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) \cdot (b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3)\\ &= a_1b_1 + a_2b_2 + a_3b_3.\end{aligned} \tag{3.11}$$

In particular

$$(\mathbf{a})^2 = |\,\mathbf{a}\,|^2 = a_1^2 + a_2^2 + a_3^2,$$

or

$$|\,\mathbf{a}\,| = (a_1^2 + a_2^2 + a_3^2)^{\frac{1}{2}}. \tag{3.12}$$

The vectorial product of **b** and **c** is obtained by applying the properties (2.10), (2.11), (2.17), (2.18) of the cross product and the properties (3.4)–(3.6) of the base:

$$\begin{aligned}\mathbf{b}\times\mathbf{c} &= (b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3)\times(c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3)\\ &= (b_2c_3 - b_3c_2)\mathbf{e}_1 - (b_1c_3 - b_3c_1)\mathbf{e}_2 + (b_1c_2 - b_2c_1)\mathbf{e}_3.\end{aligned}$$

This formula can more easily be remembered if the following symbols are introduced. For arbitrary numbers b_i, b_k, c_i, c_k by definition

$$\begin{vmatrix} b_i & b_k \\ c_i & c_k \end{vmatrix} = b_i\,c_k - b_k\,c_i \tag{3.13}$$

and

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{e}_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \mathbf{e}_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \mathbf{e}_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}. \tag{3.14}$$

(3.13) is a so-called 'second-order determinant' and (3.14) a 'third-order determinant'. For the time being these are just symbols used to denote the right-hand sides of the respective equations. Determinants are dealt with in Section 10. Taking into account (3.13) and (3.14), the vectorial product of **b** and **c** can be expressed as follows:

$$\mathbf{b}\times\mathbf{c} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} . \tag{3.15}$$

The scalar triple product of **a**, **b**, and **c**, written in the form (**abc**) = **a** · (**b**×**c**), is evaluated by use of (3.14) and (3.11):

$$(\mathbf{abc}) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix},$$

or introducing the third-order determinant defined by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix},$$

$$(\mathbf{abc}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} . \tag{3.16}$$

Note that, in view of (3.16), Theorem 2.5 can now be put in the following form. The vectors (a_1, a_2, a_3), (b_1, b_2, b_3), and (c_1, c_2, c_3) given in coordinates are coplanar (linearly dependent, cf. the remark at the end of Section 1) if and only if the determinant on the right-hand side of (3.16) formed by their coordinates is zero. Or, these vectors span the space (are linearly independent) if and only if the determinant formed by their coordinates is non-zero.

Note also that if $\mathbf{e}_1$ and $\mathbf{e}_2$ are non-parallel then they form a base in the plane that they span. By Theorem 1.4 any vector parallel to their plane can be expressed uniquely as a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$. Thus a one-to-one correspondence can be established between the set of vectors parallel to the plane determined by $\mathbf{e}_1$ and $\mathbf{e}_2$ and the ordered pair of numbers.

4. Applications in Analytical Geometry and in Mechanics

If a point O is fixed in the ordinary space, then making the point P correspond to an arbitrary vector $\mathbf{x}$, where $\mathbf{x} = \overrightarrow{OP}$, establishes a one-to-one correspondence between the set of vectors and the set of points. The vector $\mathbf{x}$ is then called the *position vector* of the point P, and the point O is called the *origin*. The position vector of the origin is the zero vector. If a right-handed orthonormal base $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is fixed then a one-to-one correspondence is established between the set of vectors and the ordered triples of numbers (the coordinate triples of the vectors). By making the ordered triple of coordinates of its position vector correspond to each point we have established a one-to-one correspondence between the set of points and the set of ordered triples. Having fixed the origin O and the base, we say that a *Cartesian orthogonal coordinate system* has been introduced. If $\overrightarrow{OP} \simeq (x_1, x_2, x_3)$ then we say that x_1, x_2, and x_3 are the *Cartesian orthogonal coordinates of the point P* (see Fig. 4.1). We denote this by simply writing $P(x_1, x_2, x_3)$.

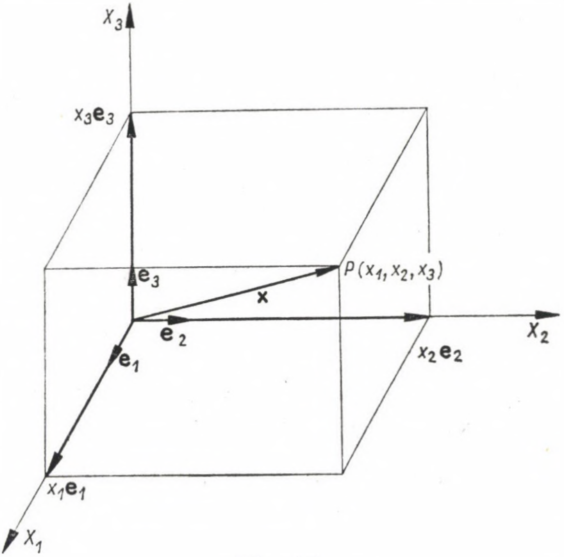


Fig. 4.1

Although the coordinate triple of a point and that of its position vector is the same, the point and its position vector are different objects. For instance, we talk about the sum and scalar multiple of vectors (even of position vectors) but not about the sum and scalar multiple of points.

The *distance* between the points $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$, i.e. the length of the straight line section AB is the length of the vector $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$. Thus, taking into account (3.9) and (3.12),

$$AB = |\overrightarrow{OB} - \overrightarrow{OA}| = [(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2]^{\frac{1}{2}}. \tag{4.1}$$

We now discuss the equations of the basic linear subsets of space, the equations of straight lines and planes. An equation with an unknown position vector is called the *vectorial equation* of a set of points if it is satisfied by the position vectors of the points of the set, and if no position vector of a point not belonging to the set satisfies it. If the position vector in the vectorial equation is given by its coordinates we have the scalar equation or system of equations of the set.

A plane is determined by a point belonging to it and a non-zero vector orthogonal to it. A non-zero vector orthogonal to the plane is called its *normal vector.*

THEOREM 4.1. The vectorial equation of the plane passing through the point A (with position vector $\mathbf{a}$) and having $\mathbf{n} \neq \mathbf{0}$ as a normal vector is

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{a}) = 0. \tag{4.2}$$

Proof. Equation (4.2) is satisfied by position vectors with terminal points on the plane; no other position vector satisfies it. If the terminal point of the position vector $\mathbf{x}$ belongs to the plane then $\mathbf{x} - \mathbf{a}$ is parallel to the plane and therefore orthogonal to $\mathbf{n}$; thus, by Theorem 2.2 the scalar product on the left-hand side of (4.2) is zero. If, conversely, (4.2) holds for some $\mathbf{x}$ then $\mathbf{x} - \mathbf{a}$ is orthogonal to $\mathbf{n}$ and therefore parallel to the plane. Since the point A belongs to the plane, the terminal point of the vector $\mathbf{x} = \mathbf{a} + (\mathbf{x} - \mathbf{a})$ is also in the plane. !

If equation (4.2) is divided by $|\mathbf{n}|$, i.e. in the equation of the plane a unit normal vector is used, the resulting equation

$$\frac{\mathbf{n}}{|\mathbf{n}|} \cdot (\mathbf{x} - \mathbf{a}) = 0 \tag{4.3}$$

is called the *normal form of the equation of the plane.*

The normal form (4.3) is useful if the distance of a point from the plane is to be determined. If the terminal point P of the position vector $\mathbf{x}$ is on the plane then (4.3) is satisfied. However, if P is not on the plane and its position vector $\mathbf{x}$ is substituted into the left-hand side of (4.3) then this expression yields the distance d between P and the plane, i.e. (see Fig. 4.2)

$$d = \pm \frac{\mathbf{n}}{|\mathbf{n}|} (\mathbf{x} - \mathbf{a}) . \tag{4.4}$$

Our plane divides the space into two half-spaces. The plus or minus sign holds in (4.4) depending on whether P is in the half-space into which the vector $\mathbf{n}$ is pointing or in the other one.

The scalar equation of the plane can be obtained by expressing the vectors in (4.2) with the help of their coordinates and evaluating the scalar product on the left-hand side according to (3.11). This calculation will be performed in the proof of the following theorem.

THEOREM 4.2. The equation of any plane can be written in the form

$$n_1x_1 + n_2x_2 + n_3x_3 + q = 0, \tag{4.5}$$

where n_1, n_2, n_3, and q are constants and, conversely, (4.5) is the equation of a plane provided that at least one of n_1, n_2, and n_3 is not zero.

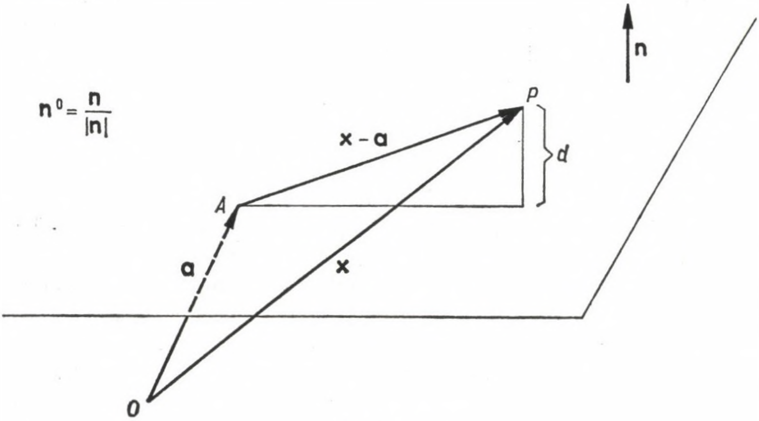


Fig. 4.2

Proof. Let a plane be given by its vectorial equation (4.2) and $\mathbf{n} \simeq (n_1, n_2, n_3)$, $\mathbf{a} \simeq (a_1, a_2, a_3)$, and $\mathbf{x} \simeq (x_1, x_2, x_3)$. Then (4.2) assumes the form

$$n_1(x_1 - a_1) + n_2(x_2 - a_2) + n_3(x_3 - a_3) = 0,$$

or the form

$$n_1x_1 + n_2x_2 + n_3x_3 + q = 0$$

where $q = -(n_1a_1 + n_2a_2 + n_3a_3)$. Conversely, we are going to show that if $n_1^2 + n_2^2 + n_3^2 > 0$, then the points (x_1, x_2, x_3) satisfying (4.5) are points of a plane. Assume, without loss of generality, that $n_1 \neq 0$. Then the unknown a_1 can be expressed, from the equation $n_1a_1 + n_2a_2 + n_3a_3 = -q$, as

$$a_1 = \frac{-q - n_2a_2 - n_3a_3}{n_1}.$$

Fixing the values of a_2 and a_3 arbitrarily, a_1 is determined by the last equation so that (4.5) assumes the form

$$n_1x_1 + n_2x_2 + n_3x_3 - n_1a_1 - n_2a_2 - n_3a_3 = 0,$$

or

$$n_1(x_1 - a_1) + n_2(x_2 - a_2) + n_3(x_3 - a_3) = 0.$$

Introducing the vectors $\mathbf{n} \simeq (n_1, n_2, n_3)$ and $\mathbf{a} \simeq (a_1, a_2, a_3)$ the last equation is the equation of the plane passing through the terminal point of **a** and having **n** as a normal vector. !

There is yet another way to describe the set consisting of the points of a plane. Let **u** and **v** be non-parallel vectors. If our plane passes through the terminal point of the position vector **a** and is parallel to **u** and **v** then the terminal point of the position vector **x** belongs to the plane if and only if the vector $\mathbf{x} - \mathbf{a}$ is parallel to the plane spanned by **u** and **v**. By Theorem 1.4 and the remark preceding it this happens if and only if two scalar numbers λ and μ exist for which

$$\mathbf{x} - \mathbf{a} = \lambda\mathbf{u} + \mu\mathbf{v}$$

holds. Thus

$$\mathbf{x} = \mathbf{a} + \lambda\mathbf{u} + \mu\mathbf{v} \tag{4.6}$$

is the so-called *parametric vectorial equation of the plane*. This means that for arbitrary real values of the *parameters* λ and μ the terminal point of the vector **x** given by (4.6) belongs to the plane and, conversely, for the position vector **x** of any point lying in the plane two real numbers λ and μ can be found for which (4.6) holds.

If we set $\mathbf{a} \simeq (a_1, a_2, a_3)$, $\mathbf{u} \simeq (u_1, u_2, u_3)$, $\mathbf{v} \simeq (v_1, v_2, v_3)$ and $\mathbf{x} \simeq (x_1, x_2, x_3)$ then equating the respective coordinates on the left and on the right-hand side of (4.6) we get the parametric system of equations of the plane:

$$\begin{aligned} x_1 &= a_1 + \lambda u_1 + \mu v_1 \\ x_2 &= a_2 + \lambda u_2 + \mu v_2 \\ x_3 &= a_3 + \lambda u_3 + \mu v_3. \end{aligned} \tag{4.7}$$

We now discuss the description of a straight line in space. Clearly, a point on the line and any non-zero vector parallel to the line determines it. A non-zero vector parallel to a straight line is called the *direction vector* of the latter.

Theorem 4.3. *The parametric vectorial equation of the straight line*, passing through the terminal point of the position vector **a** and having the non-zero vector **v** as a direction vector, is

$$\mathbf{x} = \mathbf{a} + t\mathbf{v}, \tag{4.8}$$

i.e. for any real value of the parameter t the terminal point of the position vector **x** is on the straight line and, conversely, for any point **x** on the straight line a real number t can be found such that (4.8) is satisfied.

Proof. Since the terminal point of **a** is on the straight line, the terminal point of the position vector $\mathbf{x} = \mathbf{a} + (\mathbf{x} - \mathbf{a})$ is on the straight line if and only if $\mathbf{x} - \mathbf{a}$ is parallel to **v**. Since $\mathbf{v} \neq \mathbf{0}$, by Theorem 1.3 this happens if and only if there exists a number t for which $\mathbf{x} - \mathbf{a} = t\mathbf{v}$. !

If we set $\mathbf{a} \simeq (a_1, a_2, a_3)$, $\mathbf{v} \simeq (v_1, v_2, v_3)$, and $\mathbf{x} \simeq (x_1, x_2, x_3)$ then equating the respective coordinates on the left and on the right-hand side of (4.8) we get the parametric system of equations of the straight line:

$$\begin{aligned} x_1 &= a_1 + tv_1 \\ x_2 &= a_2 + tv_2 \\ x_3 &= a_3 + tv_3. \end{aligned} \tag{4.9}$$

In particular, if none of the coordinates of $\mathbf{v}$ is zero, i.e. $v_k \neq 0$, $(k = 1, 2, 3)$, then for any point on the line the (common) value of t can be expressed from each equation of (4.9):

$$t = \frac{x_k - a_k}{v_k}, \qquad (k = 1, 2, 3).$$

This means that in the special case under consideration the coordinates of the points of the straight line satisfy the system of equations

$$\frac{x_1 - a_1}{v_1} = \frac{x_2 - a_2}{v_2} = \frac{x_3 - a_3}{v_3}. \tag{4.10}$$

Remember that while the equation of all straight lines can be given in the form (4.9), in a fixed Cartesian orthogonal system of coordinates not all straight lines can be described by a system similar to (4.10).

We conclude this Section with two applications to mechanics.

Example 4.1. Suppose that a rigid body is rotating around a fixed axis with constant angular speed ω (it is rotating uniformly). Consider any plane section of the body by a plane containing the axis. ω is the angle of rotation of this plane in unit time, i.e. the angle between the initial and the final position of this plane section during any time interval of unit length. The velocity distribution in the rigid body, i.e. the velocity (vector) of every point of the body, is found from the so-called angular velocity. The *angular velocity* is a vector $\boldsymbol{\omega}$ whose length is the angular speed ω, whose direction is parallel to the axis, and whose sense is such that looking at the body in the direction opposite to that sense the rotation is positive (counter-clockwise). Assume that the origin O has been chosen on the axis of rotation and let P denote an arbitrary point of the body; the corresponding position vector is then $\mathbf{x} = \overrightarrow{OP}$. The velocity $\mathbf{v}$ of the point P can be expressed in terms of the angular velocity and the position vector by

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}. \tag{4.11}$$

This is because P moves along a circle which is in a plane perpendicular to the axis with the centre of the circle on the axis and the radius of the circle being $\varrho = |\mathbf{x}| \sin \varphi$, where φ is the angle of $\boldsymbol{\omega}$ and $\mathbf{x}$ (see Fig. 4.3).

Then the speed of P is $\omega\varrho = |\boldsymbol{\omega}|\,|\mathbf{x}|\sin\varphi = |\boldsymbol{\omega}\times\mathbf{x}| = |\mathbf{v}|$; the direction of the velocity is given by the tangent to the circle at P, which is perpendicular to the plane spanned by $\boldsymbol{\omega}$ and $\mathbf{x}$; since from the direction of $\boldsymbol{\omega}$ the rotation is counter-clockwise, so $\boldsymbol{\omega}$, $\mathbf{x}$, and the velocity form a right-handed system.

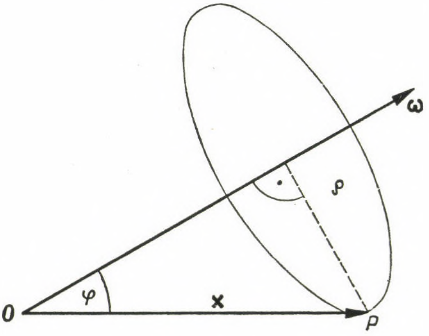


Fig. 4.3

EXAMPLE 4.2. If some small bodies which can be considered dimensionless have mass m_k ($m_k > 0$) and are situated at the points P_k ($k = 1, 2, \ldots, n$), respectively, we talk about a discrete system of masses. We say that a weight $\mathbf{g}$ is acting in space if the force given by the vector $\mathbf{g}$ is acting on any body of unit mass irrespective of its position in space. The *centroid* of our discrete system of masses is the point P for which the resultant moment (see Example 2.2) of the system, due to an arbitrary weight $\mathbf{g}$ acting in space, is zero. Denoting the position vector of P_k and the (unknown) position vector of P by $\mathbf{x}^k$ and $\mathbf{x}$, respectively, we can write

$$\sum_{k=1}^{n} (\mathbf{x}^k - \mathbf{x})\times(m_k\,\mathbf{g}) = \mathbf{0}\,,$$

or

$$\left(\sum_{k=1}^{n} m_k\,\mathbf{x}^k - \mathbf{x}\sum_{k=1}^{n} m_k\right)\times\mathbf{g} = \mathbf{0}\,.$$

Since the vector in brackets is parallel to any vector $\mathbf{g}$ it is the zero vector:

$$\sum_{k=1}^{n} m_k\,\mathbf{x}^k - \mathbf{x}\sum_{k=1}^{n} m_k = \mathbf{0}\,.$$

Hence, the position vector of the centroid is

$$\mathbf{x} = \frac{\sum_{k=1}^{n} m_k\,\mathbf{x}^k}{\sum_{k=1}^{n} m_k}\,. \tag{4.12}$$

Exercises

I.1 Prove (using vectors) that a quadrilateral is a parallelogram if and only if its diagonals divide each other into halves.

I.2 Prove (using vectors) that the medians of a triangle intersect in a single point S and that S divides each median in the ratio $1 : 2$.

I.3 What can be said about two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$, if the length of the projection of $\mathbf{b}$ onto $\mathbf{a}$ is the same as the length of the projection of $\mathbf{a}$ onto $\mathbf{b}$?

I.4 What can be said about the vectors $\mathbf{a}$ and $\mathbf{b}$ if for any scalars α and β the vectors $\alpha\mathbf{a} + \beta\mathbf{b}$ and $\beta\mathbf{a} - \alpha\mathbf{b}$ are orthogonal?

I.5 Find the angle of the vectors $\mathbf{a}$ and $\mathbf{b}$ if neither is a zero vector and

$$(\mathbf{a}\times\mathbf{b})^2 = (\mathbf{a}\cdot\mathbf{b})^2.$$

I.6 Prove (using vectors) that the altitudes of a triangle intersect in a single point.

I.7 Prove Heron's rule (using vectors). (If the sides of a triangle are a, b, c and $s = \frac{1}{2}(a + b + c)$, then the area A of the triangle

$$A = \sqrt{s(s - a)(s - b)(s - c)}\,.)$$

I.8 If three orthogonal vectors, $\mathbf{a}_1$ $\mathbf{a}_2$ and $\mathbf{a}_3$ are given in the orthonormal base $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ by $\mathbf{a}_1 = 2\,\mathbf{e}_1 + 3\,\mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{a}_2 = -\mathbf{e}_1 + 2\,\mathbf{e}_2 + 4\,\mathbf{e}_3$, $\mathbf{a}_3 = 2\,\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3$, express $\mathbf{x} = 6\,\mathbf{e}_1 + 4\,\mathbf{e}_2 + 10\,\mathbf{e}_3$ as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$.

I.9 Given two vectors $\mathbf{a} \simeq (1, -1, 2)$ and $\mathbf{b} \simeq (3, 4, -2)$, determine $\mathbf{b}$ as a sum of a vector parallel to $\mathbf{a}$ and a vector perpendicular to $\mathbf{a}$.

I.10 Given two points P_1 and P_2 and two positive numbers m, n, find the point on the line segment of P_1P_2 for which $\dfrac{P_1P}{P_2P} = \dfrac{m}{n}$.

I.11 Find the equation of the plane passing through the non-collinear points $P_1(1, -2, 3)$, $P_2(3, 4, 1)$, $P_3(2, 1, 1)$.

I.12 Find the equation of the plane that passes through the point $P(3, 4, 2)$ and contains the line $\dfrac{x_1 - 1}{2} = \dfrac{2x_2 - 5}{3} = \dfrac{x_3 - 2}{-4}$.

I.13 Find the equation of the plane that contains the parallel lines

$$\begin{aligned} x_1 &= 2t - 1 & &\text{and} & x_1 &= 2t + 7\\ x_2 &= 3t + 2 & & & x_2 &= 3t - 4\\ x_3 &= -t + 1 & & & x_3 &= -t - 2. \end{aligned}$$

I.14 Prove that the lines

$$\begin{aligned} x_1 &= 3t - 1 & &\text{and} & x_1 &= t + 6\\ x_2 &= t - 1 & & & x_2 &= 2t + \frac{11}{2}\\ x_3 &= -2t - 7 & & & x_3 &= 4t \end{aligned}$$

intersect. Find the equation of the plane that contains the lines.

I.15 Prove that the lines

$$\frac{x_1 + 1}{2} = x_2 - 1 = \frac{x_3 + 1}{2} \qquad \text{and}$$

$$\frac{x_1 + 1}{4} = -\frac{x_2 - 1}{3} = \frac{x_3 + 1}{12}$$

intersect. Determine the equation of a line which passes through the intersection point of the lines and bisects their angle.

I.16 Find the distance between the lines

$$\begin{aligned} x_1 &= t + 2 & \qquad \text{and} \qquad x_1 &= 2 \\ x_2 &= 6t - 1 & x_2 &= 2t \\ x_3 &= 2t + 1 & x_3 &= t - 1. \end{aligned}$$

I.17 Find the equation of the normal transversal (the line intersecting the given lines at right angles) of the lines

$$\begin{aligned} x_1 &= 1 + t & \qquad \text{and} \qquad x_1 &= 2 + t \\ x_2 &= 3 + 2t & x_2 &= -1 + 3t \\ x_3 &= 2t & x_3 &= t. \end{aligned}$$

I.18. Given two intersecting lines

$$\begin{aligned} x_1 &= -1 + 3t & \qquad \text{and} \qquad x_1 &= -1 + t \\ x_2 &= 2 + t & x_2 &= 2 \\ x_3 &= -2t & x_3 &= 2t \end{aligned}$$

and the plane S: $3x_1 + 7x_2 + cx_3 = 1$, find the value of the parameter c for which S is perpendicular to the plane containing the lines.

I.19 Find the point P on the axis x_1 whose respective distances from the following planes are equal:

$$\begin{aligned} 3x_1 + 2x_2 - x_3 + 7 &= 0 \\ -x_1 + 2x_2 + 3x_3 - 1 &= 0. \end{aligned}$$

CHAPTER TWO

COMPLEX ALGEBRA

5. Complex Numbers

The operations defined for vector algebra induce corresponding operations among the ordered triples of real numbers, when an orthonormal base is introduced in space. For vectors parallel to the same plane, the vector algebraic operations among these plane vectors correspond to the appropriate operations among the ordered pairs of real numbers, since such pairs represent the plane vectors in an orthonormal base in the plane. Thus, induced by the vector algebraic operations, operations such as addition, multiplication by a real number, scalar product, and vectorial product are defined in the set of ordered pairs. However, the operations so defined have certain disadvantages–the set of ordered pairs is not closed with respect to the scalar and the vectorial product. That is, a scalar product is no longer an ordered pair, but a scalar and the vectorial product is a vector which does not belong to the plane whose vectors are represented by the ordered pairs. Another disadvantage in some circumstances is that, based on operations induced by vector algebra, no division of ordered pairs can be defined. The concept of complex number, as applied to ordered pairs in whose set the operations have been defined differently, is free of these disadvantages.

We denote the set of real numbers from now on by $\mathfrak{R}$. Consider the set of ordered pairs of real numbers or, in other words, the Cartesian product of the set $\mathfrak{R}$ with itself, and denote this set by $\mathfrak{R}^2$. That is,

$$\mathfrak{R}^2 = \{(x, y) : x \in \mathfrak{R},\ y \in \mathfrak{R}\}. \tag{5.1}$$

Two elements of this set are equal (are the same) if and only if their respective coordinates are equal, i.e. $(x_1, y_1) = (x_2, y_2)$ if and only if $x_1 = x_2$ and $y_1 = y_2$. We define in the set $\mathfrak{R}^2$ *addition*, *multiplication by a real number*, and *multiplication* so that the results of these operations also belong to the set $\mathfrak{R}^2$. For arbitrary $(x_1, y_1) \in \mathfrak{R}^2$, $(x_2, y_2) \in \mathfrak{R}^2$, and $\lambda \in \mathfrak{R}$, by definition

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \tag{5.2}$$

$$\lambda(x_1, y_1) = (\lambda x_1, \lambda y_1) \tag{5.3}$$

and

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2). \tag{5.4}$$

Addition and multiplication by a scalar are defined as in vector algebra (cf. (5.2) and (5.3) with (3.9) and (3.10), respectively). Multiplication–the product of two ordered pairs–is defined by (5.4) in a way which may seem highly artificial. But we later put (5.4) into a more natural form and the advantages of definition (5.4) are then apparent.

DEFINITION 5.1. The set of ordered pairs (of real numbers) in which the operations are defined by (5.2)–(5.4) is called the *set of complex numbers* and denoted by $\mathcal{C}$; its elements are the *complex numbers*.

The operations defined in the set $\mathcal{C}$ satisfy the following operational laws ($z = (x, y)$, $z_k = (x_k, y_k)$, $(k = 1, 2, 3)$ are arbitrary complex numbers, λ and μ are arbitrary real numbers)

$$z_1 + z_2 = z_2 + z_1 \tag{5.5}$$

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3, \tag{5.6}$$

i.e. addition is commutative and associative. If the complex number $0 = (0,0)$ is added to any complex number, this does not alter the latter's value:

$$z + (0, 0) = z. \tag{5.7}$$

If the complex number $-z = (-x, -y)$ is added to z, the result is the zero complex number:

$$z + (-z) = (0, 0) = 0. \tag{5.8}$$

Multiplication by a scalar satisfies

$$\lambda(\mu z) = (\lambda\mu)z, \tag{5.9}$$

$$(\lambda + \mu)z = \lambda z + \mu z, \tag{5.10}$$

$$\lambda(z_1 + z_2) = \lambda z_1 + \lambda z_2. \tag{5.11}$$

Multiplication of complex numbers is commutative and associative:

$$z_1 z_2 = z_2 z_1, \tag{5.12}$$

and

$$z_1(z_2 z_3) = (z_1 z_2)z_3. \tag{5.13}$$

Multiplication is distributive over addition:

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3. \tag{5.14}$$

If the complex number z is multiplied by the number $(1,0)$, this does not alter the value of the former:

$$(1, 0) \cdot z = z. \tag{5.15}$$

The identities (5.5)–(5.15) are immediate consequences of the definitions (5.2)–(5.4) and of the operational laws in the set $\mathcal{R}$. As an example we prove (5.14); the proof of the rest is left to the reader.

Proof of (5.14).

$$\begin{aligned} z_1(z_2+z_3) &= (x_1, y_1)\,(x_2+x_3, y_2+y_3) \\ &= \big(x_1(x_2+x_3) - y_1(y_2+y_3),\ y_1(x_2+x_3) + x_1(y_2+y_3)\big) \\ &= \big((x_1x_2 - y_1y_2) + (x_1x_3 - y_1y_3),\ (x_1y_2 + y_1x_2) + (x_1y_3 + y_1x_3)\big) \\ &= (x_1x_2 - y_1y_2,\ x_1y_2 + y_1x_2) + (x_1x_3 - y_1y_3,\ x_1y_3 + y_1x_3) \\ &= z_1z_2 + z_1z_3.\ ! \end{aligned}$$

The *difference* of the complex numbers z_1 and z_2 is defined by

$$z_1 - z_2 = (x_1, y_1) - (x_2, y_2) = (x_1 - x_2,\ y_1 - y_2). \tag{5.16}$$

Clearly this difference is the sum of z_1 and $-z_2$.

DEFINITION 5.2. The complex number z is called the *quotient* of the complex numbers z_1 and z_2 and is denoted by $z = z_1/z_2$ if $z_1 = zz_2$.

It is easy to see that if $z_2 = 0$ and $z_1 \neq 0$ then no quotient exists and if $z_2 = z_1 = 0$ then it is not uniquely determined. On the contrary,

THEOREM 5.1. If $z_2 \neq 0$ and z_1 is arbitrary then there exists one and only one z such that $z_1 = zz_2$, i.e. the quotient of z_1 and z_2 is uniquely determined.

Proof. The uniqueness will be proved first. Assume that $z = (x, y)$ satisfies $z_1 = zz_2$, that is

$$(x_1, y_1) = (x, y)\ (x_2, y_2) = (xx_2 - yy_2,\ xy_2 + yx_2). \tag{5.17}$$

Hence

$$xx_2 - yy_2 = x_1,\ xy_2 + yx_2 = y_1. \tag{5.18}$$

Multiply the first of these equations by x_2 and the second by y_2 and add the two equations. We get

$$x(x_2^2 + y_2^2) = x_1x_2 + y_1y_2.$$

Multiply the first equation in (5.18) by $(-y_2)$ and the second by x_2 and add the two equations. We get

$$y(x_2^2 + y_2^2) = y_1x_2 - x_1y_2.$$

Since $z_2 \neq 0$, at least one of x_2 and y_2 is non-zero, i.e. $x_2^2 + y_2^2 > 0$. Thus, from the last two equations we have

$$x = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2},\ y = \frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}. \tag{5.19}$$

This means that $z = (x, y)$ is uniquely determined. Substituting (5.19) into (5.17) we see that $z = (x, y)$ is the quotient of z_1 and z_2. !

The previous theorem shows that *division* can be performed by any non-zero complex number and the result is uniquely determined. This is why we call the ordered pairs in this context (complex) *numbers.*

Now we show that the set of real numbers is in a certain sense contained in the set $\mathcal{C}$. Consider the subset $\mathcal{A}$ of $\mathcal{C}$ built up of those complex numbers whose second coordinate is zero, i.e. let

$$\mathcal{A} = \{(x, 0) : x \in \mathcal{R}\}.$$

The set $\mathcal{A}$ contains the zero (0, 0) and the number (1, 0). $\mathcal{A}$ is closed with respect to addition, subtraction, multiplication, and division defined among the complex numbers, since for arbitrary $x_1, x_2 \in \mathcal{R}$, i.e. $(x_1, 0), (x_2, 0) \in \mathcal{A}$:

$$(x_1, 0) \pm (x_2, 0) = (x_1 \pm x_2, 0) \in \mathcal{A} \tag{5.20}$$

$$(x_1, 0) \cdot (x_2, 0) = (x_1 x_2, 0) \in \mathcal{A}, \tag{5.21}$$

and if $x_2 \neq 0$, then

$$\frac{(x_1, 0)}{(x_2, 0)} = \left(\frac{x_1}{x_2}, 0\right), \tag{5.22}$$

where (5.2), (5.4), and (5.19) have been used, respectively.

There is a natural one-to-one mapping of $\mathcal{R}$ onto the set $\mathcal{A}$ by making the complex number $(x, 0) \in \mathcal{A}$ correspond to each $x \in \mathcal{R}$. It can be seen from formulae (5.20)–(5.22) that this mapping is operation-preserving. That is, $(x_1, 0)$ and $(x_2, 0)$ being the image of the real numbers x_1 and x_2, respectively, the image of $x_1 \pm x_2$, $x_1 x_2$, and x_1/x_2 (where $x_2 \neq 0$) is $(x_1, 0) \pm (x_2, 0)$, $(x_1, 0) \cdot (x_2, 0)$, and $(x_1, 0)/(x_2, 0)$, respectively. We therefore say that $\mathcal{A}$ is *isomorphic* to the set $\mathcal{R}$ of real numbers.

Further, it is easy to see that the product of an element $(x_1, 0)$ of $\mathcal{A}$ with an arbitrary complex number (x, y) is the same as if (x, y) had been multiplied by the real number x_1:

$$(x_1, 0) \cdot (x, y) = (x_1 x, x_1 y) = x_1(x, y).$$

(Applying (5.4) on the left-hand side and (5.3) on the right-hand side we get the same result in the middle.)

These facts justify the identification of the set $\mathcal{A}$ with the set $\mathcal{R}$ of real numbers and the use of the notation $x = (x, 0)$ for all $x \in \mathcal{R}$. In this terminology one can say that the set $\mathcal{C}$ of complex numbers contains the set $\mathcal{R}$ of real numbers, i.e. the former is an extension of the latter set.

There is an important geometric representation of the set $\mathcal{C}$ if, following K. F. Gauss, we introduce a Cartesian orthogonal coordinate system in the ordinary plane and make the point with coordinates (x, y) correspond to the complex number $z = (x, y)$. Obviously, this is a one-to-one mapping of the set $\mathcal{C}$ onto the plane, the latter being called in this context the *complex plane.*

Thus the complex numbers can be identified with the points of the complex plane. However, sometimes it is useful to represent the complex numbers by the position vectors of the corresponding points. Both representations will be used.

It is clear from Theorem 1.4 that an arbitrary position vector (x, y) in the complex plane can be given as a uniquely determined linear combination of the non-parallel position vectors $(1, 0)$ and $(0, 1)$, namely

$$(x, y) = x(1, 0) + y(0, 1).$$

Since multiplication by scalars (real numbers) is performed on complex numbers in the same way as on vectors given by their coordinates, the last identity can be expressed by saying that any complex number (x, y) is a unique linear combination of the complex numbers $(1, 0)$ and $(0, 1)$. According to our previous statements, the complex number $(1, 0)$ can be identified with the real number 1, i.e. $1 = (1, 0)$. The complex number $(0, 1)$ will be called the *imaginary unit* and denoted by i

$$i = (0, 1). \tag{5.23}$$

Using this notation any complex number $z = (x, y)$ can be written (uniquely) in the following form

$$z = x + iy. \tag{5.24}$$

Expression (5.24) is called the *canonical* (or *algebraic*) *form* of the complex number.

Note that the square of the imaginary unit i in (5.23) is equal to -1, since by (5.4)

$$i^2 = i \cdot i = (0, 1) \cdot (0,1) = (-1, 0) = -1. \tag{5.25}$$

Note also that i is not the only complex number having -1 as its square. Another complex number with this property is $-i$.

The method used hitherto in the introduction of complex numbers can be reversed. One can postulate (5.25) the operational laws (5.5)–(5.8), (5.12)–(5.15), and the Theorem 5.1. One has to postulate that the real numbers are contained in the set and that the set is not too broad–that is, all elements can be given in the form (5.24) where x and y are real numbers. Then the elements of this set can be given by ordered pairs and the multiplication rule (5.4)–which has been our starting point–can be deduced. This is shown in the following calculation:

$$\begin{aligned}(x_1, y_1) \cdot (x_2, y_2) &= (x_1 + iy_1) \cdot (x_2 + iy_2) \\ &= x_1x_2 + iy_1x_2 + ix_1y_2 + i^2y_1y_2 \\ &= x_1x_2 - y_1y_2 + i(x_1y_2 + y_1x_2) \\ &= (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2).\end{aligned}$$

Given the complex number $z = (x, y) = x + iy$, the real numbers x and y are called its *real part* and *imaginary part*, respectively, and they are denoted by

$$x = \operatorname{Re} z, \quad y = \operatorname{Im} z.$$

The complex numbers of the form $(0, y) = iy$, that is, those having a real part equal to zero, are called *imaginary numbers*. In the complex plane the axis x and the axis y are called the *real axis* and the *imaginary axis*, respectively. Obviously, in the representation of complex numbers in the complex plane the set of real numbers corresponds to the real axis and the set of imaginary numbers to the imaginary axis.

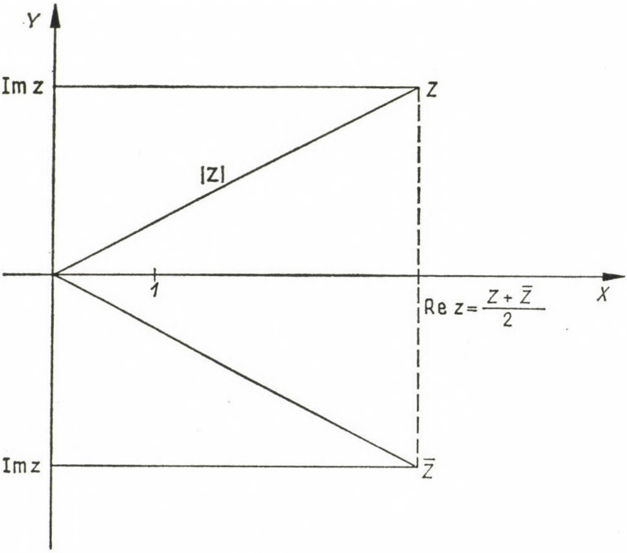


Fig. 5.1

DEFINITION 5.3. Given the complex number $z = x + iy$, the number $x - iy$ is called its *conjugate* and denoted by $\bar{z} = x - iy$.

The point $\bar{z}$ in the complex plane is the reflection of the point z in the real axis (see Fig. 5.1). Since

$$\operatorname{Re} \bar{z} = \operatorname{Re} z, \quad \operatorname{Im} \bar{z} = -\operatorname{Im} z,$$

we have

$$z + \bar{z} = 2 \operatorname{Re} z, \quad z - \bar{z} = 2i \operatorname{Im} z,$$

and hence

$$\operatorname{Re} z = \frac{1}{2}(z + \bar{z}), \quad \operatorname{Im} z = \frac{1}{2i}(z - \bar{z}). \tag{5.26}$$

It can be seen from (5.26) that the sum of a complex number and its conjugate is real, and the difference between a complex number and its conjugate is imaginary.

It is easy to see from Definition 5.3 that for arbitrary complex numbers z, z_1, and z_2

$$\bar{\bar{z}} = z, \tag{5.27}$$

i.e. the conjugate of the conjugate is the original number, and further

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2,\ \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2. \tag{5.28}$$

Identities analogous to (5.28) hold also in the more general case when the number of terms in the sum or in the product is any finite number greater than two. The proofs are left to the reader.

DEFINITION 5.4. The *modulus* (or the *absolute value*) of the complex number $z = x + iy$ is the non-negative (real) number denoted by $|z|$ and given by

$$|z| = (x^2 + y^2)^{\frac{1}{2}}.$$

From Definitions 5.3 and 5.4 it follows that $|z| = |\bar{z}|$ and

$$|z|^2 = z\bar{z} \tag{5.29}$$

(while in general the square of a non-real complex number z is not equal to the square of its modulus, i.e. $z^2 \neq |z|^2$). Formula (5.29) shows that the product of a complex number and its conjugate is a non-negative number. It is easy to see from the definitions again that

$$|\operatorname{Re} z| \leq |z|,\ |\operatorname{Im} z| \leq |z|. \tag{5.30}$$

We also have the identity

$$|z_1 z_2| = |z_1|\,|z_2|, \tag{5.31}$$

since by (5.28) and (5.29)

$$|z_1 z_2|^2 = z_1 z_2 (\overline{z_1 z_2}) = (z_1 \bar{z}_1)\ (z_2 \bar{z}_2) = |z_1|^2\,|z_2|^2.$$

The definition of the modulus accords with formula (3.12) if applied to plane vectors (whose third coordinate is zero). As a consequence, still requiring an independent proof, we have the *triangle inequality*

$$|z_1 + z_2| \leq |z_1| + |z_2|. \tag{5.32}$$

Proof of (5.32). By using (5.29), (5.28), and (5.27)

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)\ (\overline{z_1 + z_2}) = (z_1 + z_2)\ (\bar{z}_1 + \bar{z}_2) \\ &= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + z_2\bar{z}_1 \\ &= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \overline{z_1\bar{z}_2}. \end{aligned}$$

Applying the first formula in (5.26) to $z_1\bar{z}_2$ we have

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2 \operatorname{Re} (z_1\bar{z}_2).$$

Applying the first inequality in (5.30) to $z_1\bar{z}_2$ and then the formula (5.31) we get

$$\begin{aligned} | z_1 + z_2 |^2 &\leq | z_1 |^2 + | z_2 |^2 + 2 | z_1\bar{z}_2 | \\ &= | z_1 |^2 + | z_2 |^2 + 2 | z_1 | \, | z_2 | = (| z_1 | + | z_2 |)^2 . \, ! \end{aligned}$$

From (5.32) one can deduce the inequality

$$| \, | z_1 | - | z_2 | \, | \leq | z_1 \pm z_2 | \tag{5.33}$$

in the same way as (1.8) was proved.

6. The Polar Form of Complex Numbers

The addition and subtraction of complex numbers given in canonical form is easily performed. Multiplication and division are more complicated. They are more easily performed if the complex numbers are given in the form to be introduced below.

We introduce a so-called polar coordinate system in the complex plane by representing each point by its distance from the origin (the length of its position vector) and the angle between its position vector and the positive real axis; this angle is taken to be positive if it is measured counter-clockwise from the positive x axis, and negative in the opposite case (see Fig. 6.1). If the distance of the point (x, y) from the origin is denoted by r and the angle between the po-

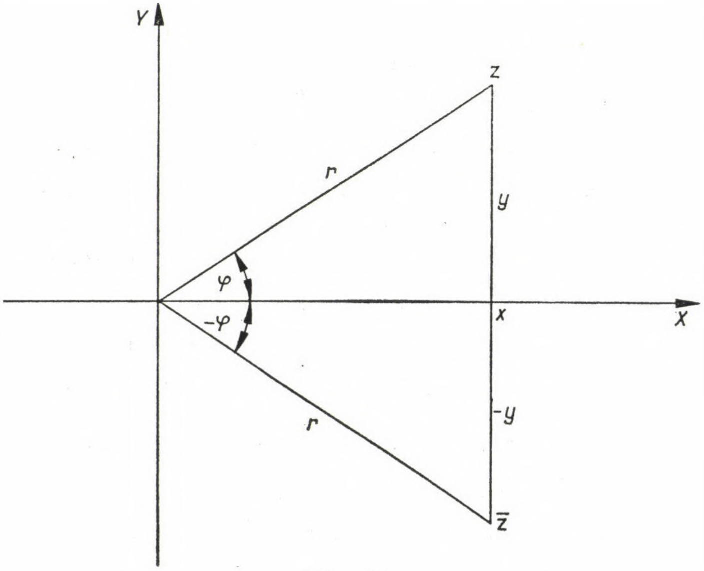


Fig. 6.1

sition vector of the point and the positive real axis is denoted by φ, then

$$x = r\cos\varphi,\ y = r\sin\varphi,$$

or, considering angles of rotation (i.e. angles larger than 2π or less than -2π),

$$x = r\cos(\varphi + 2k\pi),\ y = r\sin(\varphi + 2k\pi) \qquad k = 0, \pm 1, \pm 2, \ldots$$

From these formulae we see that the polar coordinates r and φ determine the point. Inversely, while the point determines its first polar coordinate, r, it does not determine uniquely its second polar coordinate, the angle φ. If φ is an angle corresponding to the point then any of the angles $\varphi + 2k\pi$, $(k = 0, \pm 1, \pm 2, \ldots)$—and only these—determine the same point. The only exception is the origin, whose second polar coordinate is completely arbitrary.

If the polar coordinates of the point representing the complex number $z = x + iy$ are r and φ, then

$$z = r\big(\cos(\varphi + 2k\pi) + i\sin(\varphi + 2k\pi)\big), \quad k = 0, \pm 1, \pm 2, \ldots \tag{6.1}$$

Expression (6.1) is called the *polar* (or *trigonometric*) *form* of the complex number. Since the distance of the point (x, y) from the origin is $r = (x^2 + y^2)^{\frac{1}{2}}$, the polar coordinate r in the polar form of z is the modulus of z:

$$r = |z| \geq 0.$$

The angle $\varphi + 2k\pi$ in the polar form of z is called the *argument* of z and denoted by

$$\arg z = \varphi + 2k\pi,\ (k = 0, \pm 1, \pm 2, \ldots).$$

If $z \neq 0$, then among the arguments of z there is exactly one which is not less than 0 and less than 2π. This argument is called the *principal value* of $\arg z$ and is denoted by $\operatorname{Arg} z$:

$$0 \leq \operatorname{Arg} z < 2\pi.$$

Sometimes we write instead of (6.1)

$$z = r(\cos\varphi + i\sin\varphi) \tag{6.2}$$

and assume that φ here denotes the principal value of $\arg z$. (If z is given in the form (6.1) or (6.2) we assume that $r \geq 0$.)

We know that two complex numbers both given in canonical form are equal if and only if their real parts are equal and their imaginary parts are equal. Note that, while the modulus and an argument determine the complex number, two complex numbers given in polar form

$$z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$$

and

$$z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$$

are equal if and only if their moduli are equal and their arguments differ by an integer multiple of 2π, i.e. if and only if

$$r_1 = r_2 \text{ and } \varphi_1 = \varphi_2 + 2\,k\pi, \tag{6.3}$$

where k is an integer.

If $z = r(\cos\varphi + i\sin\varphi)$, then

$$\bar{z} = r(\cos\varphi - i\sin\varphi) = r\big(\cos(-\varphi) + i\sin(-\varphi)\big);$$

hence (see Fig. 6.1)

$$\arg\bar{z} = -\arg z. \tag{6.4}$$

Now let us see how the operations of multiplication and division are performed with complex numbers given in polar form. If

$$z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1),\ z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$$

then

$$z_1 z_2 = r_1 r_2(\cos\varphi_1 + i\sin\varphi_1)\ (\cos\varphi_2 + i\sin\varphi_2)$$

$$= r_1 r_2\big(\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 + i(\sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2)\big)$$

or using the well-known trigonometric addition formulae

$$z_1 z_2 = r_1 r_2\big(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)\big). \tag{6.5}$$

Thus by (6.5) the modulus of the product is the product of the moduli, and the argument of the product is the sum of the arguments:

$$|z_1 z_2| = |z_1| \cdot |z_2|,\quad \arg(z_1 z_2) = \arg z_1 + \arg z_2. \tag{6.6}$$

If, using the previous notation, $z_2 \neq 0$, i.e. $r_2 \neq 0$, then making use of (6.4)

$$\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{z_2\bar{z}_2} = \frac{1}{r_2^2}\left[r_1 r_2\big(\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)\big)\right]$$

$$= \frac{r_1}{r_2}\big(\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)\big). \tag{6.7}$$

Thus by (6.7) the modulus of the ratio is the ratio of the moduli, and the argument of the ratio is the difference of the arguments of the numerator and the denominator:

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|},\quad \arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2. \tag{6.8}$$

If we put $z = z_1 = z_2 = r(\cos\varphi + i\sin\varphi)$ in (6.5) we get the following expression for the *square* of a complex number:

$$z^2 = r^2(\cos 2\varphi + i\sin 2\varphi).$$

Applying the same formula we may take the nth power of a complex number for arbitrary positive integer n. It is easy to prove by mathematical induction that

$$z^n = r^n(\cos n\varphi + i \sin n\varphi) \quad (n = 1, 2, 3, \ldots). \tag{6.9}$$

(6.9) is *de Moivre's formula.*

Definition 6.1. For arbitrary positive integer n the nth *root of the complex number* z is the complex number $\sqrt[n]{z}$ whose nth power is z.

Theorem 6.1. If $z = r(\cos \varphi + i \sin \varphi)$ is non-zero and n is an arbitrary positive integer, then z has exactly n different nth roots, namely

$$(\sqrt[n]{z})_k = \sqrt[n]{r}\left(\cos\frac{\varphi + 2k\pi}{n} + i \sin\frac{\varphi + 2k\pi}{n}\right) \tag{6.10}$$

$$k = 0, 1, 2, \ldots, n - 1.$$

Proof. Assume that the complex number $w = \varrho(\cos\theta + i \sin\theta)$ is an nth root of z, i.e. $w^n = z$. Then by de Moivre's formula

$$\varrho^n(\cos n\theta + i \sin n\theta) = r(\cos\varphi + i \sin\varphi).$$

Applying the conditions (6.3) for the equality of complex numbers we get from the last equation

$$\varrho^n = r, \; n\theta = \varphi + 2k\pi.$$

Hence ϱ is uniquely determined,

$$\varrho = \sqrt[n]{r} > 0,$$

and

$$\theta = \frac{\varphi + 2k\pi}{n}, \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

Thus if w is an nth root then for some integer k we have

$$w = \sqrt[n]{r}\left(\cos\frac{\varphi + 2k\pi}{n} + i \sin\frac{\varphi + 2k\pi}{n}\right).$$

Inversely, applying de Moivre's formula to the right-hand side of the last expression we see that, for arbitrary integer k, it is an nth root of z. Thus all the nth roots of z (for fixed n) are

$$(\sqrt[n]{z})_k = \sqrt[n]{r}\left(\cos\frac{\varphi + 2k\pi}{n} + i \sin\frac{\varphi + 2k\pi}{n}\right), \tag{6.11}$$

$$(k = 0, \pm 1, \pm 2, \ldots).$$

Next we show that among the (seemingly) infinite number of nth roots of z there are exactly n different numbers. We do this by showing that if two values of k in (6.11) differ by n (or by an integer multiple of n) then the corresponding nth roots are equal. Let m be an arbitrary integer and $k = m + n$, then

$$(\sqrt[n]{z})_{m+n} = \sqrt[n]{r}\left(\cos\frac{\varphi + 2(m+n)\pi}{n} + i\sin\frac{\varphi + 2(m+n)\pi}{n}\right)$$

$$= \sqrt[n]{r}\left[\cos\left(\frac{\varphi + 2m\pi}{n} + 2\pi\right) + i\sin\left(\frac{\varphi + 2m\pi}{n} + 2\pi\right)\right]$$

$$= \sqrt[n]{r}\left(\cos\frac{\varphi + 2m\pi}{n} + i\sin\frac{\varphi + 2m\pi}{n}\right) = (\sqrt[n]{z})_m .$$

Thus it is enough to consider in (6.11), at most, n successive integer values of k, e.g. those considered in (6.10). Further values of k do not yield nth roots different from these. At the same time the nth roots given in (6.10) are all different, since for any integers $0 \leq k \leq l \leq n - 1$

$$\arg(\sqrt[n]{z})_l - \arg(\sqrt[n]{z})_k = \frac{\varphi + 2l\pi}{n} - \frac{\varphi + 2k\pi}{n} = \frac{l-k}{n}2\pi ,$$

that is,

$$0 < \arg(\sqrt[n]{z})_l - \arg(\sqrt[n]{z})_k < 2\pi . !$$

As one can see from (6.10) the nth roots of the non-zero complex number $z = r(\cos\varphi + i\sin\varphi)$ are represented in the complex plane by n points on the circle of radius $\sqrt[n]{r}$ and with its centre the origin. The position vector of one of the points forms the angle φ/n with the positive real axis and the points are equally spaced on the circle, the angle of two neighbouring position vectors being equal to $2\pi/n$.

The nth roots of the complex number $1 = \cos 0 + i \sin 0$ are called the nth *roots of unity* and denoted by ω_{nk}, $(k = 0, 1, 2, \ldots, n - 1)$. By (6.10) we have

$$\omega_{nk} = (\sqrt[n]{1})_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \qquad (6.12)$$

$$(k = 0, 1, 2, \ldots, n-1).$$

In particular $\omega_{n0} = 1$ and $\omega_{n1} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. Applying de Moivre's formula (6.9)

$$\omega_{nk} = \omega_{n1}^k, \qquad (k = 0, 1, 2, \ldots, n-1). \qquad (6.13)$$

EXAMPLE 6.1. We are going to show that for $n \geq 2$ the sum of the nth roots of unity is zero. Since $1 = \omega_{n1}^n$, we have $\omega_{n1}^n - 1 = 0$ and this identity can be written in the form

$$(\omega_{n1} - 1)\ (\omega_{n1}^{n-1} + \omega_{n1}^{n-2} + \ldots + \omega_{n1} + 1) = 0.$$

However $\omega_{n1} - 1 \neq 0$, therefore

$$\omega_{n1}^{n-1} + \omega_{n1}^{n-2} + \ldots + \omega_{n1} + 1 = 0,$$

i.e. by (6.13)

$$\sum_{k=0}^{n-1} \omega_{nk} = 0\,.$$

Looking at the operations on complex numbers given in polar form we see that in the multiplication and division of two complex numbers the arguments are added and subtracted, respectively. In raising a complex number to a positive integer power, the argument is multiplied by the power index. This means that there is a striking analogy between operations with complex numbers on the one hand and multiplication, division, etc., of powers with a common base on the other. The role played by the power index in the latter case is played by the argument in the former. This similarity leads us to give complex numbers in exponential form where the argument occurs in the power index. First of all we define the imaginary power of the number e.

DEFINITION 6.2. For arbitrary real φ the *imaginary* $i\varphi$ *power of* e is defined by

$$e^{i\varphi} = \cos\varphi + i\sin\varphi. \tag{6.14}$$

(6.14) is *Euler's formula.*

We now show that the imaginary power of e as defined by Euler's formula preserves the basic property of the real exponential function expressed by the identity

$$e^{x_1}\, e^{x_2} = e^{x_1+x_2}.$$

In fact, for arbitrary real φ_1 and φ_2

$$\begin{aligned} e^{i\varphi_1} \cdot e^{i\varphi_2} &= (\cos\varphi_1 + i\sin\varphi_1) \cdot (\cos\varphi_2 + i\sin\varphi_2) \\ &= \cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2) = e^{i(\varphi_1+\varphi_2)}\,. \end{aligned}$$

It can similarly be proved that for arbitrary real φ

$$\frac{1}{e^{i\varphi}} = e^{-i\varphi}\,. \tag{6.15}$$

Applying Euler's formula, an exponential form can be given to the polar form of the complex number $z = r(\cos\varphi + i\sin\varphi)$, namely

$$z = re^{i\varphi}, \tag{6.16}$$

where, obviously, $r = |\,z\,|$ and $\varphi = \arg z$.

According to the above-mentioned properties of $e^{i\varphi}$ we can apply the formal rules for operations with powers of common base to complex numbers given in polar (exponential) form.

For $z_1 = r_1 e^{i\varphi_1}$, $z_2 = r_2 e^{i\varphi_2}$,

$$z_1 z_2 = r_1 r_2 \, e^{i(\varphi_1+\varphi_2)} \, . \tag{6.17}$$

If $z_2 \neq 0$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \, e^{i(\varphi_1-\varphi_2)} . \tag{6.18}$$

For $z = re^{i\varphi}$ and $n = 1, 2, \ldots$

$$z^n = r^n \, e^{i\, n\varphi} \tag{6.19}$$

and if $z \neq 0$, then

$$(\sqrt[n]{z})_k = \sqrt[n]{r} \, e^{i \frac{\varphi+2k\pi}{n}} , \qquad (k = 0, 1, 2, \ldots, n-1). \tag{6.20}$$

Extending (6.19) as usual to the case when the power index is zero and combining (6.15), (6.19), and (6.20) we have that, for arbitrary (not only positive) integer l and arbitrary positive integer m, such that l and m are relative prime numbers, one of the l/mth powers of the non-zero complex number $z = re^{i\varphi}$ is

$$z^{\frac{l}{m}} = r^{\frac{l}{m}} \, e^{i\varphi \frac{l}{m}}$$

and all the l/mth powers are

$$\left(z^{\frac{l}{m}}\right)_k = r^{\frac{l}{m}} \, e^{i \frac{\varphi l+2k\pi}{m}} \qquad (k = 0, 1, 2, \ldots, m-1) \, . \tag{6.21}$$

EXAMPLE 6.2. We show that the product of the nth roots of unity is equal to $(-1)^{n-1}$, $(n = 1, 2, 3, \ldots)$. Making use of (6.13) we get for the product

$$1 \cdot \omega_{n1} \cdot \omega_{n2} \ldots \omega_{n,n-1} = \omega_{n1} \cdot \omega_{n1}^2 \ldots \omega_{n1}^{n-1}$$

$$= \omega_{n1}{}^{1+2+\ldots+(n-1)} = \omega_{n1}{}^{\frac{1}{2}(n^2-n)}$$

where the addition formula of the arithmetical series has been applied. Writing ω_{n1} in the form

$$\omega_{n1} = e^{i\frac{2\pi}{n}} ,$$

we have

$$1 \cdot \omega_{n1} \cdot \omega_{n2} \cdots \omega_{n,n-1} = \left(e^{i\frac{2\pi}{n}}\right)^{\frac{1}{2}(n^2-n)} = e^{i\pi(n-1)} = (e^{i\pi})^{n-1} = (-1)^{n-1} \, . \, !$$

7. Polydomials

The introduction of complex numders arose from work on the roots of polynomials or, in other words, the solutions of higher-order algebraic equations. It is well known that the polynomial $z^2 + 1$ has no root in the set of real numbers, i.e. the equation $z^2 + 1 = 0$, or $z^2 = -1$ has no real solution. On the other hand, by Theorem 6.1 the number -1 has exactly two different complex square roots, namely $\sqrt{-1} = \pm i$. In other words the complex numbers i and $-i$ are the roots of the polynomial $z^2 + 1$.

Let $a_0, a_1, a_2, \ldots, a_n$ be fixed complex numbers (n a non-negative integer); the expression

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0, \tag{7.1}$$

where the independent variable z may assume any complex value, is called a *polynomial of degree n at most*. The numbers a_k ($k = 0, 1, \ldots, n$) are called the *coefficients*. If $a_n \neq 0$, then we say that (7.1) is a *polynomial of degree n*. If $a_n = 1$ then we say that (7.1) is a *monic polynomial* (of degree n).

The polynomial $p(z)$ is said to be *equal* (or *identically equal*) to the polynomial $q(z)$ and this relation is denoted by $p(z) \equiv q(z)$ if for all $z \in \mathcal{C}$ we have $p(z) = q(z)$.

It is easy to see that the sum and the product of polynomials (defined as usual) is again a polynomial and that the product of a polynomial of degree n (at most) with a polynomial of degree m (at most) is a polynomial of degree $n + m$ (at most).

If $p(z_0) = 0$, then we say that z_0 is a *root of the polynomial* $p(z)$ (or a *solution of the algebraic equation* $p(z) = 0$). If in (7.1) $a_n \neq 0$ and $p(z)$ is divided by a_n we get the monic polynomial $\frac{1}{a_n} p(z)$. Obviously z_0 is a root of $p(z)$ if and only if it is a root of the monic polynomial $\frac{1}{a_n} p(z)$. Thus in dealing with the roots of polynomials the considerations can be restricted to monic polynomials.

The following theorem is of basic importance, but since its proof requires tools borrowed from (real or complex) analysis the proof will be omitted (see L. V. Ahlfors: Complex Analysis, McGraw-Hill 1953, New York).

THEOREM 7.1. *(The Fundamental Theorem of Algebra).* Every polynomial of degree at least one has a root.

Another simple but important theorem is

THEOREM 7.2. *(Bézout's Theorem).* Let the polynomial $p(z)$ be of degree n and $n \geq 1$; the number z_0 is a root of $p(z)$ if and only if $p(z)$ can be factorized in the form

$$p(z) = (z - z_0)\, q(z) \tag{7.2}$$

where $q(z)$ is a polynomial of degree $n - 1$.

Proof. Assume $p(z)$ in the form (7.1) where $n \geq 1$ and $a_n \neq 0$. Condition (7.2) is clearly sufficient, since if it holds then

$$p(z_0) = (z_0 - z_0)\, q(z_0) = 0.$$

Conversely, assume that $p(z_0) = 0$. Then

$$p(z) = p(z) - p(z_0) = \sum_{k=0}^{n} a_k z^k - \sum_{k=0}^{n} a_k z_0^k = \sum_{k=1}^{n} a_k (z^k - z_0^k)\,. \tag{7.3}$$

However

$$z^k - z_0^k = (z - z_0)\,(z^{k-1} + z^{k-2} z_0 + z^{k-3} z_0^2 + \ldots + z_0^{k-1}),$$
$$(k = 1, 2, \ldots, n).$$

Hence the right-hand side of (7.3) can be factorized as follows:

$$p(z) = (z - z_0) \sum_{k=1}^{n} a_k (z^{k-1} + z^{k-2} z_0 + \ldots + z_0^{k-1})\,.$$

In the last expression the second factor is a polynomial of degree $n - 1$, for the highest power of z occurring in it is z^{n-1} and its coefficient a_n is non-zero. !

The previous two theorems have a number of important consequences.

Corollary 7.3. Any polynomial given in the form (7.1) with $n \geq 1$ and $a_n \neq 0$ can be factorized as follows:

$$p(z) \equiv a_n (z - z_1)\,(z - z_2) \ldots (z - z_n) \equiv a_n \prod_{k=1}^{n} (z - z_k) \tag{7.4}$$

(here $z_1, \ldots, z_n$ are not necessarily different complex numbers).

Proof. By Theorem 7.1 $p(z)$ has a root z_1 and by Theorem 7.2 $p(z) = (z - z_1) q_1(z)$, where $q_1(z)$ is a polynomial of degree $n - 1$ in which the coefficient of z^{n-1} is a_n. If $n - 1 > 0$ then Theorems 7.1 and 7.2 can be applied to the polynomial $q_1(z)$. Combining the result with the previous one we get $p(z) = (z - z_1)\,(z - z_2) q_2(z)$, where $q_2(z)$ is a polynomial of degree $n - 2$ in which the coefficient of z^{n-2} is a_n. Proceeding similarly, after having completed n steps we get (7.4). !

Corollary 7.4. A polynomial of degree n, with $n \geq 1$, has at most n different roots.

Proof. By the previous Corollary the polynomial can be factorized into the form (7.4). The right-hand side of (7.4) is zero if and only if a factor there is zero. Hence z_0 is a root of the polynomial if and only if $z_0 = z_k$ for some $k = 1, 2, \ldots, n$. !

COROLLARY 7.5. The polynomials

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

and

$$q(z) = b_n z^n + b_{n-1} z^{n-1} + \ldots + b_1 z + b_0$$

of degree n at most are equal if and only if

$$a_k = b_k, \qquad (k = 0, 1, 2, \ldots, n). \tag{7.5}$$

Proof. Condition (7.5) is obviously sufficient, since if it holds the two polynomials are the same. Conversely, we now prove that if the polynomials are identically equal then the corresponding coefficients are equal. Actually, it is sufficient to assume that there are $n + 1$ different values of z at which the values of the two polynomials are equal. (7.5) follows already from this assumption. Let $z_0, z_1, z_2, \ldots, z_n$ be such that $z_i \neq z_k$, if $i \neq k$, and $p(z_k) = q(z_k)$, $k = 0, 1, 2, \ldots, n$. Then the polynomial

$$p(z) - q(z) \equiv (a_n - b_n)\, z^n + \ldots + (a_1 - b_1)\, z + a_0 - b_0,$$

which is of degree n at most, has $n + 1$ different roots. By Corollary 7.4 this implies that $p(z) - q(z)$ is of degree 0, i.e.

$$a_k - b_k = 0, \qquad (k = 1, 2, \ldots, n)$$

and $p(z) - q(z) \equiv a_0 - b_0$. But

$$0 = p(z_k) - q(z_k) = a_0 - b_0. \; !$$

COROLLARY 7.6. The factorization of the polynomial $p(z)$ of degree $n (n \geq 1)$ given in (7.4) is unique (apart from the order of the factors).

Proof. The method of mathematical induction (for n) will be used. Let first $n = 1$ and assume that for some b_1 and ζ_1

$$a_1 z + a_0 \equiv a_1(z - z_1) \equiv b_1(z - \zeta_1).$$

$a_1 \neq 0$ and clearly $b_1 \neq 0$. The polynomial is zero at z_1, thus $\zeta_1 = z_1$. Dividing the identity above by $z - z_1 = z - \zeta_1$ (for $z \neq \zeta_1$) we get $b_1 = a_1$.

Assume now that every polynomial of degree $n - 1$ has a unique factorization similar to (7.4) and that for the polynomial $p(z)$ of degree n

$$p(z) \equiv a_n \prod_{k=1}^{n} (z - z_k) \equiv b_n \prod_{k=1}^{n} (z - \zeta_k)\,. \tag{7.6}$$

Clearly $b_n \neq 0$. If $z = z_n$ is substituted into (7.6) we get

$$b_n \prod_{k=1}^{n} (z_n - \zeta_k) = 0\,.$$

Hence, there is an integer m $(1 \leq m \leq n)$ for which $\zeta_m = z_n$. Let us divide the identity (7.6) by $z - z_n = z - \zeta_m$. Then for $z \neq z_n = \zeta_m$

$$a_n \prod_{k=1}^{n-1} (z - z_k) = \frac{a_n \prod\limits_{k=1}^{n} (z - z_k)}{z - z_n}$$

and

$$b_n \prod_{\substack{k=1 \\ k \neq m}}^{n} (z - \zeta_k) = \frac{b_n \prod\limits_{\substack{k=1 \\ k \neq m}}^{n} (z - \zeta_k)}{z - \zeta_m} .$$

The polynomials of degree $n - 1$ on the left-hand sides are equal for all $z \neq z_n$. Thus, by the remark made in the proof of the previous corollary they are identically equal:

$$a_n \prod_{k=1}^{n-1} (z - z_k) \equiv b_n \prod_{\substack{k=1 \\ k \neq m}}^{n} (z - \zeta_k) .$$

But the polynomials in the last identity are already of degree $n - 1$, thus, by the assumption of induction the same factors occur on both sides. !

Corollary 7.7. Any polynomial $p(z)$ of degree at least one (given, say, in the form (7.1), $n \geq 1$, $a_n \neq 0$) has a unique factorization of the form

$$p(z) \equiv a_n (z - z_1)^{m_1} (z - z_2)^{m_2} \dots (z - z_r)^{m_r}$$

$$\equiv a_n \prod_{k=1}^{r} (z - z_k)^{m_k}, \tag{7.7}$$

where $z_i \neq z_j$, if $i \neq j$, i.e. z_k $(k = 1, 2, \dots, r)$ are the different roots of the polynomial, m_k $(k = 1, 2, \dots, r)$ are positive integers, $\sum\limits_{k=1}^{r} m_k = n$ and $1 \leq r \leq n$.

Proof. An immediate consequence of Corollaries 7.3 and 7.6. !

The positive integer m_k occurring in (7.7) is called the *multiplicity* of the root z_k.

An important conclusion of the foregoing results can be summarized in the following way: *if $n \geq 1$ a polynomial of degree n has exactly n roots provided that different roots are counted with their respective multiplicities* (i.e. the number of times they occur).

There are some important relations between some symmetric expressions of the roots and the coefficients of the polynomial. If the polynomial (7.1) is of degree n, i.e. $a_n \neq 0$ with $n \geq 1$ and z_k

$(k = 1, 2, \ldots, n)$ are the (not necessarily different) roots of the polynomial (a multiple root taken into account as many times as its multiplicity), then

$$\begin{aligned}
&z_1 + z_2 + \ldots + z_n = -\frac{a_{n-1}}{a_n} \\
&z_1z_2 + z_1z_3 + \ldots + z_{n-1}z_n = \frac{a_{n-2}}{a_n} \\
&z_1z_2z_3 + \ldots + z_{n-2}z_{n-1}z_n = -\frac{a_{n-3}}{a_n} \\
&\cdot \quad \cdots \quad \cdot \quad \cdot \\
&z_1z_2 \ldots z_m + \ldots + z_{n-m+1}z_{n-m+2} \ldots z_n = (-1)^m \frac{a_{n-m}}{a_n} \\
&\cdot \quad \cdots \quad \cdot \quad \cdot \\
&z_1z_2 \ldots z_n = (-1)^n \frac{a_0}{a_n}. \qquad (7.8)
\end{aligned}$$

Formulae (7.8) are called *Vietà's formulae* (in the mth formula ($m = 1, 2, \ldots, n$) all combinations of class m of the n roots are to be taken, the roots belonging to a combination are to be multiplied, and all these products are to be summed).

Vietà's formulae can easily be deduced from the identity

$$\sum_{k=0}^{n} a_k z^k = a_n \prod_{k=1}^{n} (z - z_k)$$

if the multiplication is performed on the right-hand side and the coefficients on both sides are compared, applying Corollary 7.5.

Everything that has been discussed up to this point in this Section concerns polynomials with complex coefficients in general. The results are particularly valid for polynomials with real coefficients since we consider the set $\mathfrak{R}$ as a subset of $\mathfrak{C}$. At the same time, there are some important special features of polynomials with real coefficients and we deal with these in the rest of the Section.

THEOREM 7.8. If the coefficients of the polynomial (7.1) are real numbers, i.e. Im $a_k = 0$, $(k = 0, 1, 2, \ldots, n)$, and z_0 is a root then the conjugate of z_0 is a root too and the multiplicities of z_0 and $\bar{z}_0$ are equal.

Proof. Taking the conjugate of the equality $p(z_0) = 0$, applying (5.27), and taking into consideration that $\bar{a}_k = a_k$ $(k = 0, 1, 2, \ldots, n)$ we get

$$0 = \overline{p(z_0)} = \overline{\sum_{k=0}^{n} a_k z_0^k} = \sum_{k=0}^{n} \overline{a_k z_0^k} = \sum_{k=0}^{n} a_k \bar{z}_0^k = p(\bar{z}_0)\,;$$

thus, $\bar{z}_0$ is indeed a root. By Bézout's Theorem (Theorem 7.2), $p(z)$ is divisible by the product $(z - z_0)\,(z - \bar{z}_0)$. But

$$(z - z_0)\,(z - \bar{z}_0) = z^2 - (z_0 + \bar{z}_0)\,z + z_0\bar{z}_0 \tag{7.9}$$

and because of the first identity of (5.25) and (5.28), this is a polynomial with real coefficients. Thus the polynomial $q_2(z)$ of degree $n - 2$ defined by

$$p(z) = (z - z_0)\;(z - \bar{z}_0)\,q_2(z)$$

is also a polynomial with real coefficients. If z_0 is a root of $q_2(z)$ then $\bar{z}_0$ is also a root of the latter polynomial, i.e. if z_0 is a double root of $p(z)$ then so is $\bar{z}_0$. Proceeding further we find that the multiplicities of z_0 and $\bar{z}_0$ are the same. !

COROLLARY 7.9. If (7.1) is a polynomial of degree $n (n \geq 1, a_n \neq 0)$ with real coefficients, then it has the following unique real factorization

$$p(z) = a_n \prod_{k=1}^{r} (z - z_k)^{m_k} \cdot \prod_{h=1}^{s} (z^2 + b_h z + c_h)^{l_h} \tag{7.10}$$

where z_k are the real roots with respective multiplicities m_k $(k = 1, 2, \ldots, r;\ z_i \neq z_j$ if $i \neq j)$, b_h, c_h are real numbers $(h = 1, 2, \ldots s;\ (b_i, c_i) \neq (b_j, c_j)$ if $i \neq j)$, the quadratic polynomials $z^2 + b_h z + c_h$ have no real roots $(h = 1, 2, \ldots, s)$, m_k and l_h are nonnegative integers,

$$\sum_{k=1}^{r} m_k + 2 \sum_{h=1}^{s} l_h = n$$

and

$$0 \leq r \leq n,\ 0 \leq 2s \leq n.$$

Proof. This is an immediate consequence of Corollary 7.7 and Theorem 7.8 if at each pair of conjugate complex roots (7.9) is taken into account. !

Exercises

II.1 Find all values of z for which

$$|\,z\,| = 1 \text{ and } \left|\frac{1 - z}{1 + z^2}\right| = 1\,.$$

II.2 Prove that if $|\,z_1\,| = 1$, then for any complex number $z_2 \neq -z_1$

$$\left|\frac{z_1 + z_2}{1 + \bar{z}_1 \cdot z_2}\right| = 1\,.$$

II.3 Prove that for no positive integer n has the equation

$$(1 + iz)^n = \frac{1 + i\sqrt{3}}{\sqrt{3} + i}$$

a real solution.

II.4 Show that the complex number

$$z = \frac{1+i}{\sqrt[3]{-2+2i}}$$

has only three different positive integer powers.

II.5 Show that for $n = 3, 4, \ldots$ the sum of the squares of the nth roots of unity is zero. Is this true for $n = 2$?

II.6 Find the roots of the equation

$$z^2 - (2+3i)z - 1 + 3i = 0.$$

II.7 Determine the vertices of a regular hexagon on the complex plane if its centre is at the point $3 + 3i$ and one of its vertices is at $4 + 3i$.

II.8 Prove that the equation of any circle or straight line in the complex plane has the form

$$\alpha z \cdot \bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0,$$

where α and γ are real constants while β may be a complex constant.

II.9 Find real values for u and v such that the polynomial

$$z^4 - 2z^3 + 3z^2 + uz + v$$

is divisible by $z^2 - 1$.

II.10 Show that if the real parts of all the roots of a polynomial

$$p(z) = z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

with real coefficients are negative, then all the coefficients a_k, $k = 0, 1, \ldots n - 1$, need to be positive.

II.11 Determine the multiplicity of the solution i in the equation

$$z^6 + 5z^4 + 7z^2 + 3 = 0.$$

II.12 a) Determine the monic polynomial of least positive degree having 3 as a double and $1 + i$ as a simple root.

b) Determine the monic polynomial with real coefficients of least positive degree having 3 as a double and $1 + i$ as a simple root.

Chapter Three

MATRIX ALGEBRA

8. Elementary Operations with Matrices

Example 8.1. Assume that a company is processing m kinds of raw materials and is turning out n kinds of products. Denote by a_{ik} the amount of the ith raw material used in producing one unit of the kth product ($i = 1, 2, \ldots, m$; $k = 1, 2, \ldots, n$).

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2k} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{ik} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mk} & \cdots & a_{mn} \end{bmatrix} \tag{8.1}$$

is the specific raw material consumption table of the company. The entries in the ith row of the table $a_{i1}, a_{i2}, \ldots, a_{in}$ express the information concerning the ith kind of raw material (namely the amount consumed in producing a unit amount of the different products). The entries in the kth column of the table $a_{1k}, a_{2k}, \ldots, a_{mk}$ contain the information concerning the kth kind of products (namely the amounts of the different raw materials needed to produce one unit of the product).

Tables similar to that of the previous example occur often in different fields of mathematics, natural sciences, technology, economics, etc. In this chapter we study such arrays of data in detail, and define operations on such arrays.

Definition 8.1. If $m \cdot n$ (real or complex) numbers a_{ik} ($i = 1, 2, \ldots, \ldots, m$, $k = 1, 2, \ldots, n$) are given then the array (8.1) is called a (rectangular) *matrix of type* $m \times n$ (or an *$m \times n$ matrix*).

Matrices will be denoted in most cases by capital letters–for example, (8.1) will be denoted by A and the abbreviation $A = [a_{ik}]$ will also be used. The numbers a_{ik} are called the *elements* of the matrix A. The n-tuple $[a_{i1}, a_{i2}, \ldots, a_{in}]$ and the m-tuple $[a_{1k}, a_{2k}, \ldots, a_{mk}]$ are called the ith *row* and the kth *column* of the matrix A, respectively.

Irrespective of the actual letters used, the first subscript always denotes the row and the second the column of the element. Thus the $m \times n$ matrix $B = [b_{rs}]$ means that $r = 1, 2, \ldots, m$; $s = 1, 2, \ldots, n$ and

$$B = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ b_{m1} & b_{m2} & \ldots & b_{mn} \end{bmatrix}.$$

Definition 8.2. We say that the $m \times n$ matrix $A = [a_{ik}]$ and the $p \times q$ matrix $B = [b_{ik}]$ are *equal*, $A = B$, if they are of the same type, i.e. $m = p$, $n = q$ and if elements with equal subscripts are equal. That is,

$$a_{ik} = b_{ik}, \quad i = 1, 2, \ldots, m; \; k = 1, 2, \ldots, n.$$

Some special types of matrices play important parts in the discussion and in the applications. The $m \times n$ matrix whose every element is zero will be called the ($m \times n$) *zero matrix* and denoted by O.

An $m \times 1$ matrix (a matrix having a single column) will be called an (m-dimensional) *column vector* (or a *column matrix*) and denoted by a lower-case letter. At the same time, the second subscript denoting the column will be omitted from the elements. Thus

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_m \end{bmatrix}$$

is an m-dimensional column vector, sometimes also written in the form $a = \text{col}\,[a_1, a_2, \ldots, a_m]$ for convenience.

A $1 \times n$ matrix (a matrix having a single row) will be called an (n-dimensional) *row vector* (or *row matrix*) and denoted by a lower-case letter with a dash. At the same time the first subscript denoting the row will be omitted from the elements. Thus

$$b' = [b_1, b_2, \ldots, b_n]$$

is an n-dimensional row vector.

Matrices of type $n \times n$, i.e. matrices having an equal number of rows and columns, will be called *nth-order quadratic (square) matrices* (or *matrices of order n*). If $A = [a_{ik}]$ is a matrix of order n then we say that the elements $a_{11}, a_{22}, \ldots, a_{nn}$ form its *main diagonal*. The matrix $D = [d_{ik}]$ of order n is called a *diagonal matrix* if all elements outside the main diagonal are equal to zero, i.e. if $d_{ik} = 0$, for $i \neq k$. The diagonal matrix of order n whose elements in the main diagonal are all equal to one is called the *unit matrix* of order n and denoted by I, or I_n. Thus

$$I_n = \begin{array}{c} \begin{array}{ccccc} \underbrace{1} & \underbrace{2} & \underbrace{3} & & \underbrace{n} \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \ldots & 1 \end{array}\right] \end{array} \begin{array}{l} \\ (1 \\ (2 \\ (3 \\ \vdots \\ (n \end{array} \tag{8.2}$$

The so-called *Kronecker symbol* δ_{ik} $(i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, n)$ is defined as

$$\delta_{ik} = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{if } i \neq k\,. \end{cases} \tag{8.3}$$

Applying the Kronecker symbol, we can write the unit matrix in the form $I = [\delta_{ik}]$.

We start now to define some operations with matrices. The first operation to be defined involves a single matrix only.

DEFINITION 8.3. If $A = [a_{ik}]$ is an $m \times n$ matrix then the $n \times m$ matrix $A' = [a'_{ki}]$ is called the *transpose* of A if

$$a'_{ki} = a_{ki}, \qquad (i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n).$$

Obviously the transpose of A is obtained from A by interchanging its rows with its columns, i.e. the first row of A' is the first column of A, the second row of A' is the second column of A, etc. If, in particular, A is a square matrix then A' is obtained by reflecting the elements in the main diagonal (the elements of the main diagonal are unchanged).

The transpose of the m-dimensional column vector

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_m \end{bmatrix}$$

fs the m-dimensional row vector $a' = [a_1, a_2, \ldots, a_m]$. The transpose of the n-dimensional row vector $b' = [b_1, b_2, \ldots, b_n]$ is the n-dimensional column vector

$$(b')' = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix}.$$

Definition 8.4. If $A = [a_{ik}]$ is an $m \times n$ matrix (with complex elements, in general) then the $m \times n$ matrix $\bar{A} = [\bar{a}_{ik}]$, where $\bar{a}_{ik}$ is the complex conjugate of a_{ik}, is called the *conjugate* of A. The $n \times m$ matrix $A^* = [a^*_{ki}]$ is called the *adjoint* (or the *conjugate transpose*) of A if $A^* = \bar{A}'$, i.e. if $a^*_{ki} = \bar{a}_{ik}$, $(i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n)$.

If the elements of A are real numbers then, obviously, $A^* = A'$. For the transpose of the transpose of $A = [a_{ik}]$, i.e. for $(A')' = [a''_{ik}]$, we have by Definition 8.3 $a''_{ik} = a'_{ki} = a_{ik}$, $(i = 1, 2, \ldots, m;$ $k = 1, 2, \ldots, n)$. That is,

$$(A')' = A. \tag{8.4}$$

Similarly, it is easy to see that

$$(A^*)^* = A. \tag{8.5}$$

Definition 8.5. If $A = [a_{ik}]$ is an $m \times n$ matrix then the $m \times n$ matrix $-A = [-a_{ik}]$ is called its *opposite*.

Definition 8.6. The square matrix A with real elements is called *symmetric* if $A' = A$. The square matrix B with complex elements is called *hermitian* if $B^* = B$.

Obviously, a real square matrix $A = [a_{ik}]$ of order n is symmetric (a complex square matrix $B = [b_{ik}]$ of order n is hermitian) if and only if the elements standing in symmetric positions with respect to the main diagonal are equal (are complex conjugate), i.e. if and only if $a_{ik} = a_{ki}$ $(b_{ik} = \bar{b}_{ki})$ for $i, k = 1, 2, \ldots, n$.

Definition 8.7. The real square matrix A is called *skew-symmetric* if $A' = -A$. The complex square matrix B is called *skew-hermitian* if $B^* = -B$.

Obviously, a real square matrix $A = [a_{ik}]$ of order n is skew-symmetric (a complex square matrix $B = [b_{ik}]$ of order n is skew-hermitian) if and only if $a_{ik} = -a_{ki}$ $(b_{ik} = -\bar{b}_{ki})$ for $i, k = 1, 2, \ldots, n$. In particular, from $a_{ii} = -a_{ii}$ $(b_{ii} = -\bar{b}_{ii})$ it follows that in a real skew-symmetric matrix all elements in the main diagonal are zero (in a complex skew-hermitian matrix all elements in the main diagonal are imaginary numbers).

We now discuss operations with two matrices.

Definition 8.8. If $A = [a_{ik}]$ and $B = [b_{ik}]$ are both $m \times n$ matrices then the $m \times n$ matrix $S = [s_{ik}]$ is called their *sum* and is denoted by $A + B$ if

$$s_{ik} = a_{ik} + b_{ik}, \quad (i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n).$$

Note that only matrices of equal type can be added and addition is performed by adding the corresponding elements.

The addition of matrices satisfies the following operational laws (A, B, and C are arbitrary matrices of the same type):

$$A + B = B + A, \tag{8.6}$$

$$(A + B) + C = A + (B + C), \tag{8.7}$$

i.e. addition is commutative and associative. This follows at once from Definition 8.8 and from the fact that addition in the set of real or complex numbers is commutative and associative (see (5.5) and (5.6)).

If A is an arbitrary matrix and O the zero matrix of the same type then, obviously,

$$A + O = A. \tag{8.8}$$

From Definitions 8.5 and 8.8 it follows that the sum of a matrix and its opposite is the zero matrix:

$$A + (-A) = O. \tag{8.9}$$

DEFINITION 8.9. If $A = [a_{ik}]$ is a matrix of type $m\times n$, and λ is a real or complex number, then the $m\times n$ matrix $S = [s_{ik}]$ for which $s_{ik} = \lambda a_{ik}$ $(i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n)$ is called the *λ multiple of A* and is denoted by $S = \lambda A = A\lambda$.

For arbitrary matrices A and B of the same type and scalars λ and μ we have

$$\lambda(\mu A) = (\lambda\mu)A, \tag{8.10}$$

$$(\lambda + \mu)A = \lambda A + \mu A, \tag{8.11}$$

$$\lambda(A + B) = \lambda A + \lambda B. \tag{8.12}$$

(8.10)–(8.12) are simple consequences of Definitions 8.9 and 8.8 and the corresponding formulae (5.8)–(5.10). Clearly

$$1A = A, \quad (-1)A = -A. \tag{8.13}$$

EXAMPLE 8.2. Assume that a preparatory plant is processing m kinds of raw materials and is turning out n kinds of semi-processed articles. Let the $m\times n$ matrix $A = [a_{ik}]$ be the specific raw material consumption table of the preparatory plant (see Example 8.1). Assume further that an assembly plant is processing the n kinds of semi-processed articles produced by the preparatory plant, and that the assembly plant is turning out p kinds of products. Let the $n\times p$ matrix $B = [b_{kl}]$ be the specific semi-processed article consumption table of the assembly plant. What is the amount of the ith raw material needed to produce one unit of the lth product $(i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p)$? We need the amount $a_{i1}, a_{i2}, \ldots, a_{in}$ of the ith raw material to produce one unit of the first, the second, up to the nth semi-processed article, respectively. At the same time we need the amount $b_{1l}, b_{2l}, \ldots, b_{nl}$ of the first, the second, up to the nth semi-processed article to produce one unit of the lth product. Thus the amount v_{il} of the ith raw material consumed in producing one unit of the lth product is

$$v_{il} = a_{i1} b_{1l} + a_{i2} b_{2l} + \ldots + a_{in} b_{nl}.$$

The $m \times p$ matrix $V = [v_{il}]$ is the specific raw material consumption table of the assembly plant.

Keeping in mind the previous example we define multiplication of matrices as follows:

DEFINITION 8.10. Let $A = [a_{ik}]$ be an $m \times n$ and $B = [b_{kl}]$ be an $n \times p$ matrix; the $m \times p$ matrix $V = [v_{il}]$ for which

$$v_{il} = \sum_{k=1}^{n} a_{ik} b_{kl}, \quad (i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p)$$

is called the *product* of A and B and is denoted by $V = AB$.

Note that the product $V = AB$ (in this order) is defined only if the number of columns in A is equal to the number of rows in B or, in other words, the number of elements in a row of A is equal to the number of elements in a column of B. This is because, in evaluating the v_{il} element of V, the ith row of A is to be combined with the lth column of B.

It is worth considering some special cases of matrix multiplication. The product of two nth order square matrices is again an nth order square matrix.

The product of the $m \times n$ matrix $A = [a_{ik}]$ and the n-dimensional $(n \times 1)$ column vector $x = [x_k]$ is the m-dimensional $(m \times 1)$ column vector $b = [b_i]$ where

$$b_i = \sum_{k=1}^{n} a_{ik} x_k, \quad (i = 1, 2, \ldots, m).$$

In detail:

$$\begin{bmatrix} a_{11} & a_{12} \ldots & a_{1n} \\ a_{21} & a_{22} \ldots & a_{2n} \\ \cdot & \cdot \quad \ldots & \cdot \\ a_{m1} & a_{m2} \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \cdot \qquad \cdot \qquad \ldots \qquad \cdot \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

The product of the n-dimensional $(1 \times n)$ row vector $a' = [a_i]$ and the n-dimensional $(n \times 1)$ column vector $b = [b_i]$ is the 1×1 matrix $[v]$ where $v = \sum_{i=1}^{n} a_i b_i$. In detail:

$$a'b = [a_1, a_2, \ldots, a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = [a_1 b_1 + a_2 b_2 + \ldots + a_n b_n] = [v]. \qquad (8.14)$$

(8.14) is sometimes called the *inner product* of a' and b.

There is a natural one-to-one mapping of the set of real numbers $\mathcal{R}$ (complex numbers $\mathcal{C}$) onto the set $\{[r] : r \in \mathcal{R}\}$ ($\{[z] : z \in \mathcal{C}\}$) of 1×1 matrices by making the matrix $[r]$ ($[z]$) correspond to each $r \in \mathcal{R}$ ($z \in \mathcal{C}$). It is easy to see that this mapping is operation preserving–that is, the sum and the product of two 1×1 matrices corresponding to given numbers are equal to the matrices corresponding to the sum and the product of these numbers, respectively. We express this by saying that the set of real (complex) 1×1 matrices is *isomorphic* to the set $\mathcal{R}$ ($\mathcal{C}$). Justified by this isomorphism we write instead of $[v]$ simply v.

The product of the m-dimensional ($m \times 1$) column vector $a = [a_i]$ and the n-dimensional ($1 \times n$) row vector $b' = [b_k]$ is the $m \times n$ matrix $V = [v_{ik}]$, where $v_{ik} = a_i b_k$, $(i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n)$. In detail:

$$ab' = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} [b_1, b_2, \ldots, b_n] = \begin{bmatrix} a_1b_1 & a_1b_2 & \ldots & a_1b_n \\ a_2b_1 & a_2b_2 & \ldots & a_2b_n \\ \cdot & \cdot & \cdots & \cdot \\ a_mb_1 & a_mb_2 & \ldots & a_mb_n \end{bmatrix}. \tag{8.15}$$

(8.15) is sometimes called the *outer product* (or *dyade*) of a and b'.

It is easy to see that for the mth-order unit matrix I_m, the $m \times n$ matrix A, and the nth-order unit matrix I_n

$$I_m A = A, \quad AI_n = A. \tag{8.16}$$

We also have, for the $p \times m$ zero matrix $O_{p\times m}$ the $m \times n$ matrix A, and the $n \times q$ zero matrix $O_{n\times q}$

$$O_{p\times m} A = O_{p\times n}, \quad AO_{n\times q} = O_{m\times q} \tag{8.17}$$

where, of course, $O_{p\times n}$ and $O_{m\times q}$ are the zero matrices of type denoted by the subscript. (8.16) and (8.17) express the facts that any matrix multiplied by the unit matrix stays unaltered and that any matrix multiplied by the zero matrix becomes the zero matrix, respectively.

If A is of type $m \times n$ and B is of type $n \times p$ then AB can be formed but, in general, BA is not defined. The latter product exists if and only if $m = p$. In this case AB is of type $m \times m$, i.e. a square matrix of order m, and BA is of order n, i.e. a square matrix of order n. If $m \neq n$ then AB and BA cannot be equal. But even in the only case left, where $m = n = p$, or in other words if A and B are nth-order matrices, in general

$$AB \neq BA.$$

EXAMPLE 8.3. If

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

then

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

thus $AB \neq BA$.

If, in particular, A and B are square matrices of the same order and $AB = BA$ then we say that A and B are *commutable.* It is easy to see that two diagonal matrices of the same order are always commutable. A square matrix is always commutable with itself. We use the notation $A^2 = AA$.

EXAMPLE 8.4. We give a necessary and sufficient condition for the commutability of a second-order matrix A and its transpose A'. If

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \text{then} \quad A' = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix},$$

$$AA' = \begin{bmatrix} a_{11}^2 + a_{12}^2 & a_{11}a_{21} + a_{12}a_{22} \\ a_{11}a_{21} + a_{12}a_{22} & a_{21}^2 + a_{22}^2 \end{bmatrix},$$

$$A'A = \begin{bmatrix} a_{11}^2 + a_{21}^2 & a_{11}a_{12} + a_{21}a_{22} \\ a_{11}a_{12} + a_{21}a_{22} & a_{12}^2 + a_{22}^2 \end{bmatrix}.$$

AA' is equal to $A'A$ if and only if the corresponding elements are equal, that is, if and only if

$$\begin{aligned} a_{11}^2 + a_{12}^2 &= a_{11}^2 + a_{21}^2 \\ a_{22}^2 + a_{21}^2 &= a_{22}^2 + a_{12}^2 \\ a_{11}a_{21} + a_{12}a_{22} &= a_{11}a_{12} + a_{21}a_{22}. \end{aligned}$$

The first and second equations hold if and only if $a_{12} = \pm a_{21}$. If $a_{12} = a_{21}$, i.e. if A is symmetric, then the third equation automatically holds. If $a_{12} = -a_{21} \neq 0$, then the third equation holds if and only if $a_{11} = a_{22}$. Thus A and A' are commutable if and only if either A is symmetric ($A = A'$) or the elements in the main diagonal are equal and the elements outside the main diagonal are opposite.

Matrix multiplication satisfies the following operational laws. For the matrices $A = [a_{ik}]$, $B = [b_{ik}]$, $C = [c_{ik}]$ of type $m \times n$, $n \times p$, and $p \times q$, respectively,

$$(AB)C = A(BC), \tag{8.18}$$

i.e. multiplication is associative.

Proof of (8.18). Since $U = AB = [u_{ik}]$ is of type $m \times p$ and $V = BC = [v_{ik}]$ is of type $n \times q$, both UC and AV are of type $m \times q$. We have $u_{ih} = \sum_{l=1}^{n} a_{il}b_{lh}$ and $v_{lk} = \sum_{h=1}^{p} b_{lh}c_{hk}$; thus the corresponding elements on the left-hand side and on the right-hand side of (8.18) are

$$\sum_{h=1}^{p} u_{ih}c_{hk} = \sum_{h=1}^{p} \left(\sum_{l=1}^{n} a_{il}b_{lh} \right) c_{hk}$$

and

$$\sum_{l=1}^{n} a_{il}v_{lk} = \sum_{l=1}^{n} a_{il} \left(\sum_{h=1}^{p} b_{lh}c_{hk} \right) = \sum_{l=1}^{n} \sum_{h=1}^{p} a_{il}b_{lh}c_{hk},$$

respectively. Interchanging the order of summation in the last expression, we get

$$\sum_{l=1}^{n} a_{il}\, v_{lk} = \sum_{h=1}^{p} \left(\sum_{l=1}^{n} a_{il}\, b_{lh} \right) c_{hk} = \sum_{h=1}^{p} u_{ih}\, c_{hk} \,.\, !$$

For the $m \times n$ matrix $A = [a_{ik}]$ and the $n \times p$ matrices $B = [b_{ik}]$ and $C = [c_{ik}]$,

$$A(B + C) = AB + AC. \tag{8.19}$$

Proof of (8.19). By Definitions 8.10 and 8.8 formula (8.19) means that

$$\sum_{l=1}^{n} a_{il}(b_{lk} + c_{lk}) = \sum_{l=1}^{n} a_{il}\, b_{lk} + \sum_{l=1}^{n} a_{il}\, c_{lk} \,,$$

$$(i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, p)$$

and this is obviously true. !

For the $m \times n$ matrices A and B and the $n \times p$ matrix C,

$$(A + B)C = AC + BC. \tag{8.20}$$

The proof of (8.20) is analogous to the proof of (8.19) and is omitted. (8.19) and (8.20) must be stated separately, since matrix multiplication is not commutative. These formulae mean that multiplication is distributive over addition.

For an arbitrary number λ, and matrices A and B of type $m \times n$ and $n \times p$, respectively,

$$\lambda(AB) = (\lambda A)B = A(\lambda B). \tag{8.21}$$

The proof of (8.21) is left to the reader.

It is also easy to prove that the transpose of a sum of matrices is equal to the sum of the transposes, i.e. if A and B are matrices of the same type then

$$(A + B)' = A' + B'. \tag{8.22}$$

Let $A = [a_{ik}]$ and $B = [b_{ik}]$ be matrices of type $m \times n$ and $n \times p$, respectively, then

$$(AB)' = B'A', \tag{8.23}$$

i.e. the transpose of a product is equal to the product of the transposes taken in inverted order.

Proof of (8.23). Since $U = AB = [u_{ik}]$ is of type $m \times p$, $U' = (AB)' = [u'_{ik}]$ is of type $p \times m$. The product of the $p \times n$ matrix $B' = [b'_{il}]$ and the $n \times m$ matrix $A' = [a'_{lk}]$ is also a matrix of type $p \times m$. By Definitions 8.10 and 8.3 the element in the ith row, kth column on the right-hand side of (8.23) is

$$\sum_{l=1}^{n} b'_{il}\, a'_{lk} = \sum_{l=1}^{n} a_{kl}\, b_{li} = u_{ki} = u'_{ik} \,.\, !$$

9. The Linear Space of the Ordered n-tuples

The *Cartesian product* $\mathcal{A}\times\mathcal{B}$ of two sets $\mathcal{A}$ and $\mathcal{B}$ is the set of ordered pairs (a, b) where a is taken from $\mathcal{A}$ and b from $\mathcal{B}$:

$$\mathcal{A}\times\mathcal{B} = \{ (a, b) : a \in \mathcal{A},\ b \in \mathcal{B}\}.$$

We may take, analogously, the Cartesian product of more than two sets.

The Cartesian product of n copies of the set of real numbers $\mathcal{R}$ (the set of complex numbers $\mathcal{C}$) will be denoted by $\mathcal{R}^n$ (respectively $\mathcal{C}^n$), i.e. $\mathcal{R}^n$ is the set of ordered n-tuples of real numbers ($\mathcal{C}^n$ is the set of ordered n-tuples of complex numbers):

$$\mathcal{R}^n = \{(x_1, x_2, \ldots, x_n) : x_k \in \mathcal{R},\ k = 1, 2, \ldots, n\}$$
$$(\mathcal{C}^n = \{(z_1, z_2, \ldots, z_n)\ : z_k \in \mathcal{C},\ k = 1, 2, \ldots, n\})\ .$$

We denote the set of n-dimensional real (complex) column vectors by $\mathcal{Q}^n$ and the set of n-dimensional real (complex) row vectors by $\mathcal{S}^n$. There is an obvious one-to-one correspondence between the set $\mathcal{R}^n$ ($\mathcal{C}^n$) and the set $\mathcal{Q}^n$, and a similar one-to-one correspondence between the set $\mathcal{R}^n$ ($\mathcal{C}^n$) and the set $\mathcal{S}^n$ where to each $(x_1, \ldots, x_n) \in \mathcal{R}^n$ ($\in \mathcal{C}^n$) corresponds the column vector col $[x_1, \ldots, x_n] \in \mathcal{Q}^n$ in the first case and the row vector $[x_1, \ldots, x_n] \in \mathcal{S}^n$ in the second.

We have defined some matrix algebraic operations with elements of the set $\mathcal{Q}^n$ and the set $\mathcal{S}^n$, namely, the operations of addition and the multiplication by a real or complex number. The result of these operations is again an element of the set $\mathcal{Q}^n$, respectively $\mathcal{S}^n$. Let us define these operations among the elements of $\mathcal{R}^n$ ($\mathcal{C}^n$) correspondingly, i.e. let

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1,\ x_2 + y_2, \ldots, x_n + y_n)$$

and

$$\lambda(x_1, x_2, \ldots, x_n) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n),$$

where $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are arbitrary elements of $\mathcal{R}^n$ ($\mathcal{C}^n$) and λ is an arbitrary real (complex) number.

Obviously the sets $\mathcal{R}^n$ ($\mathcal{C}^n$), $\mathcal{Q}^n$, $\mathcal{S}^n$ are equivalent *(isomorphic)* with respect to these operations. In this Section we shall consider the set $\mathcal{Q}^n$ of column vectors but the results are equally valid for the set $\mathcal{S}^n$ or $\mathcal{R}^n$ ($\mathcal{C}^n$).

In the set $\mathcal{Q}^n$, Definitions 8.8 and 8.9 induce the operations of addition and multiplication by a scalar number. In Section 8 we proved that these operations satisfy certain operational laws. Namely, for any x, y, $z \in \mathcal{Q}^n$

$$x + y \in \mathcal{Q}^n,$$
$$x + y = y + x, \tag{9.1}$$

$$(x + y) + z = x + (y + z), \tag{9.2}$$

$$x + O = x, \tag{9.3}$$

where $O = \text{col}\,[0, 0, \ldots, 0]$, the zero column matrix. Further,

$$x + (-x) = O. \tag{9.4}$$

For any α, $\beta \in \mathfrak{R}$ (or $\in \mathfrak{C}$)

$$\alpha x \in \mathfrak{Q}^n,$$

$$\alpha(x + y) = \alpha x + \alpha y, \tag{9.5}$$

$$(\alpha + \beta)x = \alpha x + \beta x, \tag{9.6}$$

$$\alpha(\beta x) = (\alpha\beta)x, \tag{9.7}$$

$$1x = x. \tag{9.8}$$

Note that the identities (9.1)–(9.8) are similar to those valid in ordinary vector algebra ((1.2)–(1.5) and (1.10)–(1.12)).

Let $x_k \in \mathfrak{Q}^n$, $\lambda_k \in \mathfrak{R}$ (or $\lambda_k \in \mathfrak{C}$) ($k = 1, 2, \ldots, m$); the column vector $\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m \in \mathfrak{Q}^n$ is called a *linear combination of* $x_1, \ldots, x_m$. If $\lambda_1 = \lambda_2 = \ldots = \lambda_m = 0$, then $O = \lambda_1 x_1 + \ldots + \lambda_m x_m$ is the *trivial linear combination* of $x_1, \ldots, x_m$.

In accordance with the remark made at the end of Section 1 (generalizing it) we define linear dependence and independence for the set of column vectors.

DEFINITION 9.1. We say that $x_1, x_2, \ldots, x_m \in \mathfrak{Q}^n$ are *linearly dependent* if there exist $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathfrak{R}$ (or $\mathfrak{C}$) such that $|\lambda_1| + |\lambda_2| + \ldots + |\lambda_m| > 0$ and

$$\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m = O.$$

In other words $x_1, \ldots, x_m$ are linearly dependent if a non-trivial linear combination of them is zero. (The condition $|\lambda_1| + \ldots + |\lambda_m| > 0$ excludes exactly the trivial case.) If $x_1, \ldots, x_m$ are linearly dependent then at least one of the λ_k-s is non-zero ($k = 1, 2, \ldots, \ldots, m$), and so at least one of these column vectors can be expressed as a linear combination of the rest.

DEFINITION 9.2. We say that $x_1, \ldots, x_m \in \mathfrak{Q}^n$ are *linearly independent* if they are not linearly dependent, i.e. if only their trivial linear combination is zero, or if the equality

$$\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m = O$$

implies

$$\lambda_1 = \lambda_2 = \ldots = \lambda_m = 0.$$

We now prove some theorems connected with linear dependence and independence. These theorems lead us to the determination of the maximal number of linearly independent column vectors in $\mathfrak{Q}^n$. At the same time we shall see how is it possible to represent all elements of $\mathcal{Q}^n$ if a sufficient number of linearly independent column vectors are known.

THEOREM 9.1. If one of $x_1, \ldots, x_m \in \mathcal{Q}^n$ is the zero column vector then these column vectors are linearly dependent.

Proof. Let for instance $x_1 = O$. Then choosing $\lambda_1 \neq 0$, $\lambda_2 = \lambda_3 = \ldots \ldots = \lambda_m = 0$, we have

$$\lambda_1 O + 0\, x_2 + \ldots + 0\, x_m = O. \,!$$

THEOREM 9.2. If $x_1, \ldots, x_m \in \mathcal{Q}^n$ are linearly dependent and $x_{m+1}, \ldots, x_p \in \mathcal{Q}^n$ are arbitrary column vectors, then $x_1, \ldots, x_m$, $x_{m+1}, \ldots, x_p$ are also linearly dependent.

Proof. By the assumption there exist numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that $|\lambda_1| + \ldots + |\lambda_m| > 0$ and $\lambda_1 x_1 + \ldots + \lambda_m x_m = O$. Choosing $\lambda_{m+1} = \ldots = \lambda_p = 0$, we have $|\lambda_1| + \ldots + |\lambda_m| + |\lambda_{m+1}| + \ldots \ldots + |\lambda_p| > 0$ and

$$\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m + 0\, x_{m+1} + \ldots + 0\, x_p = O. \,!$$

COROLLARY 9.3. If $x_1, \ldots, x_p \in \mathcal{Q}^n$ are linearly independent then any non-empty subset of $\{x_1, x_2, \ldots, x_p\}$ is also linearly independent.

THEOREM 9.4. In $\mathcal{Q}^n$ there exist n linearly independent column vectors.

Proof. Let

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \begin{array}{l} (1 \\ (2 \\ (3 \\ \vdots \\ (n \end{array} \tag{9.9}$$

or, applying the Kronecker symbol (8.3), $e_k = [\delta_{ik}]$, $(k = 1, 2, \ldots, n)$. Assume that, for some $\lambda_1, \lambda_2, \ldots, \lambda_n$, there holds $\sum_{i=1}^{n} \lambda_i e_i = O$. Applying Definitions 8.9 and 8.8 this means that

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

i.e. $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$. Thus by Definition 9.2 $e_1, \ldots, e_n$ are linearly independent. !

Theorem 9.5. *Any $x \in \mathcal{Q}^n$ can be expressed as a unique linear combination of the column vectors (9.9).*

Proof. Let $x = \operatorname{col}[x_1, x_2, \ldots, x_n]$. Since

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ \vdots \\ 0 \end{bmatrix} + \ldots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \ldots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

that is,

$$x = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n; \tag{9.10}$$

x is indeed a linear combination of $e_1, \ldots, e_n$. We now show that the decomposition (9.10) of x is uniquely determined. Assume that

$$x = y_1 e_1 + y_2 e_2 + \ldots + y_n e_n$$

and subtract the latter equality from (9.10). We get

$$(x_1 - y_1)\, e_1 + (x_2 - y_2)\, e_2 + \ldots + (x_n - y_n)\, e_n = O.$$

But by Theorem 9.4 the column vectors $e_1, \ldots, e_n$ are linearly independent. Thus $x_1 - y_1 = x_2 - y_2 = \ldots = x_n - y_n = 0$, or $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$. !

Definition 9.3. We say that the column vectors $a_1, a_2, \ldots, a_m \in \mathcal{Q}^n$ form a *base* of $\mathcal{Q}^n$ if they are linearly independent and any element of $\mathcal{Q}^n$ can be expressed as a linear combination of $a_1, \ldots, a_m$ (cf. Definition 3.1 and formula (3.1)).

Theorem 9.6. *If $a_1, a_2, \ldots, a_m$ form a base of $\mathcal{Q}^n$ then any element of $\mathcal{Q}^n$ has a unique decomposition with respect to this base.*

Proof. The proof is analogous to that part of the proof of Theorem 9.5 which follows the formula (9.10), and is thus left to the reader. !

By Theorem 9.4 and Theorem 9.5 the column vectors (9.9) form a base of $\mathcal{Q}^n$. We now show that each base of $\mathcal{Q}^n$ consists of exactly n column vectors, i.e. in Definition 9.3 necessarily $m = n$. Note that a definition equivalent to Definition 9.3 would be to call a set of n linearly independent column vectors a base of $\mathcal{Q}^n$.

Theorem 9.7. *If $b_1, b_2, \ldots, b_r \in \mathcal{Q}^n$ are linearly independent then $r \leq n$ and $n - r$ column vectors can be chosen from among the column vectors (9.9) so that $b_1, b_2, \ldots, b_r, e_{i_{r+1}}, e_{i_{r+2}}, \ldots, e_{i_n}$ form a base of $\mathcal{Q}^n$ $(i_l \neq i_k,\ l \neq k)$.*

Proof. We show that a column vector from the set (9.9) can be exchanged for b_1, so that b_1 and the remaining $n - 1$ column vectors of (9.9) form a base; then another column vector of (9.9) can be exchanged for b_2 so that b_1, b_2, and the remaining $n - 2$ vectors of (9.9) form a base, etc. Provided that this has been proved the inequality $r \leq n$ follows. The latter statement will be proved first.

Assume that the vectors b_k can be brought into the base $e_1, e_2, \ldots, e_n$ one by one by deleting from the base one vector e_i at each step. Assume then that $r > n$. After n steps all vectors (9.9) are omitted and $b_1, b_2, \ldots, b_n$ form a base of $\mathfrak{Q}^n$. Thus any element of $\mathfrak{Q}^n$, in particular b_{n+1} $(n + 1 \leq r)$, can be expressed as a linear combination of $b_1, b_2, \ldots, b_n$

$$b_{n+1} = \lambda_1 b_1 + \lambda_2 b_2 + \ldots + \lambda_n b_n$$

that is,

$$-\lambda_1 b_1 - \lambda_2 b_2 - \ldots - \lambda_n b_n + 1 b_{n+1} = 0,$$

but this contradicts the assumption that $b_1, b_2, \ldots, b_n, b_{n+1}$ are linearly independent (see Corollary 9.3).

We start to prove now that a vector of (9.9) can be exchanged for b_1 so that b_1 and the remaining $n - 1$ vectors of (9.9) form a base. Since $e_1, e_2, \ldots, e_n$ form a base, we have for some numbers $\lambda_{11}, \lambda_{12} \ldots, \lambda_{1n}$ that

$$b_1 = \lambda_{11} e_1 + \lambda_{12} e_2 + \ldots + \lambda_{1n} e_n. \tag{9.11}$$

At least one of λ_{1k} $(k = 1, 2, \ldots, n)$ is non-zero, since otherwise (9.11) would mean $b_1 = O$ and by Theorem 9.1 $b_1, b_2, \ldots, b_r$ could not be linearly independent. We may assume without loss of generality that $\lambda_{11} \neq 0$. Thus we get from (9.11)

$$e_1 = \frac{1}{\lambda_{11}} b_1 + \left(-\frac{\lambda_{12}}{\lambda_{11}}\right) e_2 + \ldots + \left(-\frac{\lambda_{1n}}{\lambda_{11}}\right) e_n .$$

Since any element of $\mathfrak{Q}^n$ can be expressed as a linear combination of $e_1, \ldots, e_n$ and e_1 is a linear combination of $b_1, e_2, \ldots, e_n$, any element of $\mathfrak{Q}^n$ can be expressed as a linear combination of $b_1, e_2, \ldots, e_n$. We show indirectly that these vectors are linearly independent. Assume, contrary to this statement, that there exist numbers $\mu_{11}, \mu_{12}, \ldots, \mu_{1n}$ not all of which are zero such that

$$\mu_{11} b_1 + \mu_{12} e_2 + \ldots + \mu_{1n} e_n = O. \tag{9.12}$$

If $\mu_{11} = 0$, then at least one of $\mu_{12}, \mu_{13}, \ldots, \mu_{1n}$ must be different from zero, so

$$\mu_{12} e_2 + \ldots + \mu_{1n} e_n = O$$

would hold where on the left-hand side we have a non-trivial linear combination. This contradicts the fact that $e_2, \ldots, e_n$ are linearly independent. Thus $\mu_{11} \neq 0$, and we get from (9.12)

$$b_1 = 0\, e_1 + \left(-\frac{\mu_{12}}{\mu_{11}}\right) e_2 + \ldots + \left(-\frac{\mu_{1n}}{\mu_{11}}\right) e_n . \tag{9.13}$$

Taking into account that, by Theorem 9.5, b_1 has a unique decomposition by the vectors $e_1, \ldots, e_n$, and comparing (9.11) with (9.13) we get $\lambda_{11} = 0$, which is a contradiction. Thus we have proved that $b_1, e_2, \ldots, e_n$ form a base in $\mathfrak{Q}^n$.

We continue by showing that one of $e_2, \ldots, e_n$ can be exchanged for b_2 to get another base. In the base $b_1, e_2, \ldots, e_n$ we have a decomposition for b_2:

$$b_2 = \lambda_{21}b_1 + \lambda_{22}e_2 + \ldots + \lambda_{2n}e_n. \tag{9.14}$$

At least one of $\lambda_{21}, \lambda_{22}, \ldots, \lambda_{2n}$ is non-zero, otherwise we should have $b_2 = O$. If $\lambda_{22} = \ldots = \lambda_{2n} = 0$, then we should have $\lambda_{21} \neq 0$ and since $b_2 - \lambda_{21}b_1 = 0$ would follow, b_1 and b_2 would be linearly dependent, contrary to the assumption. Thus not all of $\lambda_{22}, \lambda_{23}, \ldots, \lambda_{2n}$ are zero. We may assume without loss of generality that $\lambda_{22} \neq 0$. Proceeding as in the previous step we get that $b_1, b_2, e_3, \ldots, e_n$ form a base.

Proceeding further we may similarly fetch the column vectors $b_3, b_4, \ldots$ into the base at the expense of an equal number of column vectors (9.9). We have shown already that as a consequence $r \leq n$. Finally, it follows that $b_1, b_2, \ldots, b_r$ and the remaining $n - r$ column vectors of (9.9) form a base in $\mathfrak{Q}^n$. !

Theorem 9.7 has important consequences.

Corollary 9.8. Any set of $n + 1$ column vectors taken from $\mathfrak{Q}^n$ is linearly dependent. (The maximal number of linearly independent n-dimensional column vectors is n.)

Corollary 9.9. If $b_1, b_2, \ldots, b_n \in \mathfrak{Q}^n$ are linearly independent, then they form a base (i.e. any column vector of $\mathfrak{Q}^n$ can be expressed as a uniquely determined linear combination of $b_1, \ldots, b_n$).

Corollary 9.10. Any base of $\mathfrak{Q}^n$ consists of n column vectors.

Proof. It follows from Theorem 9.7 that no base in $\mathfrak{Q}^n$ may possess more than n base vectors. Assume now that $b_1, b_2, \ldots, b_r$ form a base and $r < n$. Then by Theorem 9.7 these column vectors plus $n - r$ column vectors of (9.9), say, $b_1, \ldots, b_r, e_{r+1}, \ldots, e_n$, form a base too. But since $b_1, \ldots, b_r$ form a base, the column vector e_{r+1} can be written

$$e_{r+1} = \lambda_1 b_1 + \ldots + \lambda_r b_r,$$

that is,

$$\lambda_1 b_1 + \ldots + \lambda_r b_r + (-1)\, e_{r+1} = O.$$

Thus $b_1, b_2, \ldots, b_r, e_{r+1}$ are linearly dependent and this contradicts the fact that $b_1, \ldots, b_r, e_{r+1}, \ldots, e_n$ form a base. Since r cannot be greater than n, we have $r = n$. !

We say that $\mathfrak{Q}^n$ is an *n-dimensional linear space*, or that the n-dimensional column vectors form an n-dimensional linear space (cf. Section 17).

In what follows, for example in the theory of systems of linear equations, some subsets of the linear space $\mathcal{Q}^n$ will play an important role. These are the subsets which are closed with respect to addition and multiplication by a scalar.

Definition 9.4. The subset $\mathcal{H}^m$ of the space $\mathcal{Q}^n$ is called an m-*dimensional subspace* if for any $x, y \in \mathcal{H}^m$, and $\alpha \in \mathcal{R}$ (or $\alpha \in \mathcal{C}$) we have $(x + y) \in \mathcal{H}^m$, $(\alpha x) \in \mathcal{H}^m$, and if in $\mathcal{H}^m$ there exist m linearly independent column vectors, but any set of column vectors in $\mathcal{H}^m$ consisting of more than m elements is linearly dependent.

The following facts are self-evident: the zero vector O is an element of any subspace of $\mathcal{Q}^n$; the dimension m of a subspace of $\mathcal{Q}^n$ cannot be greater than n; if $\mathcal{H}^n$ is an n-dimensional subspace of $\mathcal{Q}^n$, then $\mathcal{H}^n = \mathcal{Q}^n$ (see Corollary 9.9).

The set of all linear combinations of $a_1, a_2, \dots, a_m \in \mathcal{Q}^n$ is called the *subset spanned by* $a_1, \dots, a_m$.

Definition 9.5. Let $\mathcal{H}^m \in \mathcal{Q}^n$ be an m-dimensional subspace of $\mathcal{Q}^n$; we say that the column vectors $a_1, a_2, \dots, a_m \in \mathcal{H}^m$ form a *base of the subspace* $\mathcal{H}^m$ if they are linearly independent.

Theorem 9.11. If $a_1, \dots, a_m \in \mathcal{H}^m$ form a base of the m-dimensional subspace $\mathcal{H}^m$ then any column vector $x \in \mathcal{H}^m$ can be uniquely expressed as a linear combination of $a_1, \dots, a_m$.

Proof. It follows from Definition 9.4 that $a_1, \dots, a_m, x$ are linearly dependent, i.e. there exist $\lambda_1, \lambda_2, \dots, \lambda_m, \mu$ such that $|\lambda_1| + \dots + |\lambda_m| + |\mu| > 0$ and

$$\lambda_1 a_1 + \dots + \lambda_m a_m + \mu x = O.$$

If $\mu = 0$, then at least one of λ_k $(k = 1, 2, \dots, m)$ would be non-zero and hence $a_1, \dots, a_m$ would be linearly dependent, contrary to the assumption. Thus $\mu \neq 0$ and

$$x = \left(-\frac{\lambda_1}{\mu}\right) a_1 + \left(-\frac{\lambda_2}{\mu}\right) a_2 + \dots + \left(-\frac{\lambda_m}{\mu}\right) a_m .$$

The proof that this decomposition is uniquely determined follows the second part of the proof of Theorem 9.5 and is therefore omitted. !

Theorem 9.12. If $a_1, \dots, a_m \in \mathcal{Q}^n$ are linearly independent then the subset $\mathcal{H}^m$ spanned by $a_1, \dots, a_m$ is an m-dimensional subspace of $\mathcal{Q}^n$ (with $a_1, \dots, a_m$ as a base).

Proof. The subset

$$\mathcal{H}^m = \{\lambda_1 a_1 + \dots + \lambda_m a_m : (\lambda_1, \dots, \lambda_m) \in \mathcal{R}^m \text{ (or } \mathcal{C}^m)\}$$

is a subspace since, clearly, the sum of two linear combinations of $a_1, \dots, a_m$ and the scalar multiple of a linear combination are again linear combinations of $a_1, \dots, a_m$. We have to prove that $\mathcal{H}^m$ is

m-dimensional. Obviously, $\mathcal{H}^m$ is at least m-dimensional. Any column vector in $\mathcal{H}^m$ can be expressed as a linear combination of the linearly independent column vectors $a_1, \ldots, a_m$ and this decomposition is uniquely determined (see the second part of the proof of Theorem 9.5). Thus, if $b_1, b_2, \ldots, b_r \in \mathcal{H}^m$ are linearly independent, then the proof of Theorem 9.7 can be repeated, $a_1, \ldots, a_m$ playing the role of $e_1, \ldots, e_n$. Thus $r \leq m$. !

Corollary 9.13. Given $a_1, \ldots, a_p \in \mathcal{Q}^n$; if the maximal number of linearly independent column vectors among $a_1, \ldots, a_p$ is r $(\leq p)$ then $a_1, \ldots, a_p$ span an r-dimensional subspace $\mathcal{H}^r$ of $\mathcal{Q}^n$.

Corollary 9.14. If $a_1, \ldots, a_p \in \mathcal{Q}^n$ span an r-dimensional subspace $\mathcal{H}^r$ and $c_1, c_2, \ldots, c_q$ are linear combinations of $a_1, \ldots, a_p$, then the maximal number of linearly independent column vectors among $c_1, \ldots, c_q$ is at most r.

Example 9.1. Consider in $\mathcal{Q}^3$ the linearly independent column vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$\begin{aligned} \mathcal{H}^2 &= \{x_1 e_1 + x_2 e_2 : (x_1, x_2) \in \mathcal{R}^2\} \\ &= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} : (x_1, x_2) \in \mathcal{R}^2 \right\} \end{aligned}$$

is the two-dimensional subspace of $\mathcal{Q}^3$ spanned by e_1 and e_2. If e_1 and e_2 are considered as the column vectors corresponding to the first two vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively, of an orthonormal base in ordinary space (see Definition 3.2), then $\mathcal{H}^2$ corresponds to the set of vectors parallel to the plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$.

Example 9.2. We show that the column vectors

$$c_{r+1} = \begin{bmatrix} c_{11} \\ c_{12} \\ \vdots \\ c_{1r} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad c_{r+2} = \begin{bmatrix} c_{21} \\ c_{22} \\ \vdots \\ c_{2r} \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad c_n = \begin{bmatrix} c_{n-r,1} \\ c_{n-r,2} \\ \vdots \\ c_{n-r,r} \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

of the space $\mathcal{Q}^n$, i.e. the column vectors $c_{r+i} = \operatorname{col}[c_{i1}, c_{i2}, \ldots, c_{ir},$

$\delta_{i1}, \delta_{i2}, \ldots, \delta_{i,n-r}]$, $(i = 1, 2, \ldots, n - r$; δ_{ik} the Kronecker symbol, c_{ik} arbitrary numbers) are linearly independent. Assume that

$$\lambda_{r+1} c_{r+1} + \lambda_{r+2} c_{r+2} + \ldots + \lambda_n c_n = O.$$

Performing the operations on the left-hand side we get

$$\begin{bmatrix} \lambda_{r+1} c_{11} + \lambda_{r+2} c_{21} + \ldots + \lambda_n c_{n-r,1} \\ \lambda_{r+1} c_{12} + \lambda_{r+2} c_{22} + \ldots + \lambda_n c_{n-r,2} \\ \cdot \qquad \cdot \qquad \cdots \qquad \cdot \\ \lambda_{r+1} c_{1r} + \lambda_{r+2} c_{2r} + \ldots + \lambda_n c_{n-r,r} \\ \lambda_{r+1} \\ \lambda_{r+2} \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Hence $\lambda_{r+1} = \lambda_{r+2} = \ldots = \lambda_n = 0$.

Note that the results of this section are valid practically without alteration in the set $\mathcal{S}^n$ of the n-dimensional row vectors and in the set $\mathcal{R}^n$ $(\mathcal{C}^n)$.

10. Determinants

If n column vectors are given in the space $\mathcal{Q}^n$ of the n-dimensional column vectors, are these column vectors linearly independent or not? A similar question may be posed in the space $\mathcal{S}^n$ of row vectors. In this section we define a real (complex) valued function on the set of real (complex) square matrices; the *determinant*, which becomes zero for a matrix if and only if the columns (and the rows) of the matrix considered as column vectors (and row vectors) are linearly dependent. In other words the so-called determinant of a square matrix is a number made to correspond to this matrix such that if the columns (and the rows) of this matrix are linearly dependent then its determinant is zero, otherwise it is not (cf. the remarks after (3.16)).

However, we have to introduce some notation, some new concepts, we have to define the determinant of a matrix, and we have to show a number of basic properties before we can proceed to the main point (see Corollary 12.7).

If the column vectors $a_k \in \mathcal{Q}^n$, $(k = 1, 2, \ldots, n)$ or the row vectors $\alpha_i' \in \mathcal{S}^n$, $(i = 1, 2, \ldots, n)$ are given we may build up a square matrix with these vectors, namely the one whose kth column or ith row is a_k or α_i', respectively. Conversely, we shall consider the columns and the rows of a matrix as column vectors and row vectors, respectively. The following notation will be used throughout this Section. If A denotes the nth-order quadratic matrix

$$A = \begin{bmatrix} a_{11} \dots a_{1k} \dots a_{1n} \\ a_{21} \dots a_{2k} \dots a_{2n} \\ \cdot \quad \cdot\cdot\cdot\cdot \quad \cdot\cdot\cdot\cdot \\ a_{i1} \dots a_{ik} \dots a_{in} \\ \cdot \quad \cdot\cdot\cdot\cdot \quad \cdot\cdot\cdot\cdot \\ a_{n1} \dots a_{nk} \dots a_{nn} \end{bmatrix}, \tag{10.1}$$

then

$$a_k = \begin{bmatrix} a_{1k} \\ \vdots \\ a_{ik} \\ \vdots \\ a_{nk} \end{bmatrix}$$

is the kth column vector of A, $\alpha'_i = [a_{i1}, \dots, a_{ik}, \dots, a_{in}]$ is the ith row vector of A, $(k = 1, 2, \dots, n; i = 1, 2, \dots, n)$ and sometimes we write A as the row vector of its column vectors or, alternatively, as the column vector of its row vectors ;

$$A = [a_1, a_2, \dots, a_k, \dots, a_n] = \begin{bmatrix} \alpha'_1 \\ \alpha'_2 \\ \vdots \\ \alpha'_i \\ \vdots \\ \alpha'_n \end{bmatrix}. \tag{10.2}$$

DEFINITION 10.1. If the ith row and the kth column are deleted from the nth-order matrix A then the $(n - 1)$th-order matrix obtained in this way is called the *submatrix corresponding to the element* a_{ik} and is denoted by A_{ik}. That is,

$$A_{ik} = \begin{bmatrix} a_{11} & \dots a_{1,k-1} & a_{1,k+1} & \dots a_{1n} \\ \cdot & \cdot\cdot\cdot\cdot & \cdot & \cdot\cdot\cdot\cdot \\ a_{i-1,1} & \dots a_{i-1,k-1} & a_{i-1,k+1} & \dots a_{i-1,n} \\ a_{i+1,1} & \dots a_{i+1,k-1} & a_{i+1,k+1} & \dots a_{i+1,n} \\ \cdot & \cdot\cdot\cdot\cdot & \cdot & \cdot\cdot\cdot\cdot \\ a_{n1} & \dots a_{n,k-1} & a_{n,k+1} & \dots a_{nn} \end{bmatrix}.$$

Similarly, if we delete the ith and the jth row and the kth and the lth column of A we get an $(n - 2)$th-order *submatrix* which is denoted by $A_{ik,jl}$.

DEFINITION 10.2. The *determinant of the first-order matrix* $[a_{11}]$ is defined as the number a_{11} and denoted by

$$\det [a_{11}] = a_{11}. \tag{10.3}$$

The *determinant of the nth-order matrix* (10.1) ($n = 2, 3, \ldots$) is defined by the formula

$$\det A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \cdot & \cdots & \cdot \\ a_{n1} & \ldots & a_{nn} \end{bmatrix} = \sum_{k=1}^{n} a_{1k}(-1)^{1+k} \det A_{1k}, \tag{10.4}$$

where A_{1k} is the submatrix corresponding to the element a_{1k} ($k = 1, 2, \ldots, n$), (and it is assumed that the determinant of ($n - 1$)th-order matrices has already been defined). The determinant of an nth-order matrix is called an nth-*order determinant*.

Thus the determinant of a quadratic matrix of arbitrary order has been defined. The determinant of a second-order matrix by Definition 10.2 is

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}(-1)^2 \det [a_{22}] + a_{12}(-1)^3 \det [a_{21}]$$
$$= a_{11}a_{22} - a_{12}a_{21}, \tag{10.5}$$

where (10.3) has been used (cf. (3.13)). The determinant of a third-order matrix is by Definition 10.2

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$
$$= a_{11}(-1)^2 \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} + a_{12}(-1)^3 \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$
$$+ a_{13}(-1)^4 \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$
$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}),$$

where (10.5) has been used, etc.

In accordance with the notation introduced at the beginning of this Section, the notation

$$\det A = \det [a_1, a_2, \ldots, a_n] = \det \begin{bmatrix} \alpha_1' \\ \alpha_2' \\ \vdots \\ \alpha_n' \end{bmatrix}$$

will also be used and the column vectors and row vectors of the matrix A will also be called the column vectors and the row vectors, respectively, of its determinant.

The determinant of the submatrix A_{ik} of the matrix (10.1), i.e. $\det A_{ik}$, will be called the *minor* of the element a_{ik} and the number

$$D_{ik} = (-1)^{i+k} \det A_{ik} \tag{10.6}$$

will be called the *cofactor* of the element a_{ik}. Applying this notation, we can write (10.4) in the form

$$\det A = \sum_{k=1}^{n} a_{1k} D_{1k} \,. \tag{10.7}$$

(10.4) or (10.7) is called the expansion of the determinant by its first row.

In the rest of the Section we prove some basic properties of determinants.

THEOREM 10.1. If the ith row vector ($i = 1, 2, \ldots, n$) of a determinant is multiplied by the scalar λ then the value of the determinant is multiplied by λ.

Proof (by mathematical induction for the order n of the determinant). The statement is, clearly, valid for $n = 1$. Assume that it is valid for all $(n - 1)$th-order determinants and consider the nth-order determinant $\det A$. If $i = 1$, then by (10.7)

$$\begin{vmatrix} \lambda a_{11} & \lambda a_{12} & \ldots & \lambda a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} = \sum_{k=1}^{n} (\lambda a_{1k})\, D_{1k} = \lambda \det A \,.$$

Let $2 \le i \le n$. Since $\det A_{1k}$ is of order $(n - 1)$, by the assumption of induction

$$\begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \lambda a_{i1} & \lambda a_{i2} & \ldots & \lambda a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{2n} & \ldots & a_{nn} \end{vmatrix} = \sum_{k=1}^{n} a_{1k}(-1)^{1+k}\,(\lambda \det A_{1k})$$

$$= \sum_{k=1}^{n} a_{1k}(\lambda D_{1k}) = \lambda \det A \,. \,!$$

THEOREM 10.2. If the ith row vector ($i = 1, 2, \ldots, n$) of $\det A$ is the sum of two row vectors, i.e. $\alpha_i' = \beta_i' + \gamma_i'$, then $\det A$ is equal to the sum of two determinants, the first of which has β_i' as its ith row and the second γ_i' as its ith row, while the rest of the rows in both coincide with the corresponding rows of $\det A$.

Proof (by mathematical induction for the order n of the determinant). The statement is, clearly, valid for $n = 1$. Assume that it is valid for all $(n - 1)$th-order determinants and consider the nth-order determinant $\det A$. If $i = 1$, then by (10.7)

$$\begin{vmatrix} b_{11}+c_{11} & b_{12}+c_{12} & \dots & b_{1n}+c_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \sum_{k=1}^{n} (b_{1k}+c_{1k})\, D_{1k}$$

$$= \sum_{k=1}^{n} b_{1k} D_{1k} + \sum_{k=1}^{n} c_{1k} D_{1k} = \begin{vmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} .$$

Let $2 \le i \le n$. Since $\det A_{1k}$ is of order $(n-1)$, by the assumption of induction

$$i) \qquad \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ b_{i1}+c_{i1} & b_{i2}+c_{i2} & \dots & b_{in}+c_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

$$= \sum_{k=1}^{n} (a_{1k}(-1)^{1+k} \begin{vmatrix} a_{21} & \dots & a_{2,k-1} & a_{2,k+1} & \dots & a_{2n} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ b_{i1} & \dots & b_{i,k-1} & b_{i,k+1} & \dots & b_{in} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ a_{n1} & \dots & a_{n,k-1} & a_{n,k+1} & \dots & a_{nn} \end{vmatrix}$$

$$+ a_{1k}(-1)^{1+k} \begin{vmatrix} a_{21} & \dots & a_{2,k-1} & a_{2,k+1} & \dots & a_{2n} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ c_{i1} & \dots & c_{i,k-1} & c_{i,k+1} & \dots & a_{in} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ a_{n1} & \dots & a_{n,k-1} & a_{n,k+1} & \dots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \cdot & \cdots & \cdot \\ b_{i1} & \dots & b_{in} \\ \cdot & \cdots & \cdot \\ a_{n1} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \cdot & \cdots & \cdot \\ c_{i1} & \dots & c_{in} \\ \cdot & \cdots & \cdot \\ a_{n1} & \dots & a_{nn} \end{vmatrix} .!$$

Theorem 10.3. If two neighbouring rows of a determinant are equal then the value of the determinant is zero.

Proof (by mathematical induction for the order n of the determinant). If $n = 2$, then obviously

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{11} & a_{12} \end{vmatrix} = a_{11}a_{12} - a_{12}a_{11} = 0 .$$

Assume that the statement is valid for all $(n-1)$th-order determinants and consider the nth-order determinant $\det A$. If the two equal rows are not the first and the second then in the expansion (10.4) all $\det A_{1k}$ are determinants of order $(n-1)$ with two equal rows. Thus by the assumption of induction each term in (10.4) is equal to zero, i.e. $\det A = 0$. If the first row is equal to the second then we may write

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{11} & a_{12} & \dots & a_{1n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \sum_{k=1}^{n} a_{1k}(-1)^{1+k} \det A_{1k} .$$

Since

$$\det A_{1k} = \sum_{l=1}^{k-1} a_{1l}(-1)^{1+l} \det A_{1k,2l} + \sum_{l=k+1}^{n} a_{1l}(-1)^{1+l-1} \det A_{1k,2l} ,$$

therefore

$$\det A = \sum_{k=1}^{n} \sum_{l=1}^{k-1} a_{1k}(-1)^{1+k} a_{1l}(-1)^{1+l} \det A_{1k,2l}$$

$$+ \sum_{k=1}^{n} \sum_{l=k+1}^{n} a_{1k}(-1)^{1+k} a_{1l}(-1)^{1+l-1} \det A_{1k,2l} .$$

We are going to show that in this expansion we may find for each term another, that is the (-1) multiple of the former; thus the right-hand side is zero. Let $1 \le p < q \le n$ be arbitrarily given. The term corresponding to $k = q$, $l = p$ is found in the first sum (where $l < k$ in each term):

$$a_{1q}(-1)^{1+q} a_{1p}(-1)^{1+p} \det A_{1q,2p} = (-1)^{q+p} a_{1q} a_{1p} \det A_{1q,2p} .$$

The term corresponding to $k = p$, $l = q$ is found in the second sum (where $k < l$ in each term):

$$a_{1p}(-1)^{1+p} a_{1q}(-1)^{q} \det A_{1p,2q} = (-1)^{1+p+q} a_{1q} a_{1p} \det A_{1p,2q} .$$

The latter term is the (-1) multiple of the former since

$$A_{1q,2p} = A_{1p,2q}$$

(both submatrices derive from A by omitting the first and second rows and the pth and qth columns). !

Theorem 10.4. The determinant of the unit matrix I_n is equal to one.

Proof. It is obvious by mathematical induction for the order n. !

THEOREM 10.5. If a row vector of the determinant is the zero row matrix then the value of the determinant is zero.

Proof. If the ith row is the zero row matrix then it can be considered as the zero multiple of an arbitrary row vector. Thus by Theorem 10.1 the determinant is the zero multiple of another determinant. !

THEOREM 10.6. If two rows of a determinant are interchanged then the value of the determinant changes to its (-1) multiple.

Proof. It is sufficient to prove the statement for the case of neighbouring rows, since the interchange of the arbitrary α_i' and α_j' rows (where, say, $1 \leq i < j \leq n$) can be performed in several steps of interchanging neighbouring rows. The ith row is brought into the jth position in $(j - i)$ steps; then the former jth row is brought into the ith position in $(j - i - 1)$ steps. If the statement is already proved for the case of neighbouring rows then after performing these $(j - i) + (j - i - 1)$ interchanges of neighbouring rows we get a determinant whose value is the $(-1)^{(j-i)+(j-i-1)} = (-1)^{2(j-i)-1} = -1$ multiple of the original determinant.

Let us interchange the ith and the $(i + 1)$th rows in $\det A$. If we can prove that the sum of the old and the new determinants is zero we have proved our statement. First, we add to this sum two determinants, each with two equal neighbouring rows (both have the value zero by Theorem 10.3). Next, we perform the addition of two pairs of determinants, each pair having all rows equal but one, by Theorem 10.2.

$$\det\begin{bmatrix}\alpha_1'\\ \alpha_2'\\ \vdots\\ \alpha_i'\\ \alpha_{i+1}'\\ \vdots\\ \alpha_n'\end{bmatrix} + \det\begin{bmatrix}\alpha_1'\\ \alpha_2'\\ \vdots\\ \alpha_{i+1}'\\ \alpha_i'\\ \vdots\\ \alpha_n'\end{bmatrix}$$

$$= \det\begin{bmatrix}\alpha_1'\\ \alpha_2'\\ \vdots\\ \alpha_i'\\ \alpha_{i+1}'\\ \vdots\\ \alpha_n'\end{bmatrix} + \det\begin{bmatrix}\alpha_1'\\ \alpha_2'\\ \vdots\\ \alpha_{i+1}'\\ \alpha_{i+1}'\\ \vdots\\ \alpha_n'\end{bmatrix} + \det\begin{bmatrix}\alpha_1'\\ \alpha_2'\\ \vdots\\ \alpha_{i+1}'\\ \alpha_i'\\ \vdots\\ \alpha_n'\end{bmatrix} + \det\begin{bmatrix}\alpha_1'\\ \alpha_2'\\ \vdots\\ \alpha_i'\\ \alpha_i'\\ \vdots\\ \alpha_n'\end{bmatrix}$$

$$= \det \begin{bmatrix} \alpha_1' \\ \alpha_2' \\ \vdots \\ \alpha_i' + \alpha_{i+1}' \\ \alpha_{i+1}' \\ \vdots \\ \alpha_n' \end{bmatrix} + \det \begin{bmatrix} \alpha_1' \\ \alpha_2' \\ \vdots \\ \alpha_i' + \alpha_{i+1}' \\ \alpha_i' \\ \vdots \\ \alpha_n' \end{bmatrix} = \det \begin{bmatrix} \alpha_1' \\ \alpha_2' \\ \vdots \\ \alpha_i' + \alpha_{i+1}' \\ \alpha_i' + \alpha_{i+1}' \\ \vdots \\ \alpha_n' \end{bmatrix} = 0,$$

where Theorem 10.2 has been applied once again to get a determinant with two equal neighbouring rows, i.e. to get zero by Theorem 10.3. !

THEOREM 10.7. If two (arbitrary) rows of a determinant are equal then the value of the determinant is zero.

Proof. If the equal rows are brought into neighbouring positions then at most the sign of the determinant is changed by the previous theorem. However, we get a determinant which has equal neighbouring rows and thus is zero by Theorem 10.3. !

THEOREM 10.8. If a scalar multiple of a row vector of a determinant is added to another row the value of the determinant does not change.

Proof. Let us multiply the ith row vector of $\det A$, that is the row vector α_i' by the scalar number λ, and add the result to the jth row vector α_j' $(i \neq j)$. Applying Theorems 10.2, 10.1 and 10.7 we get for the resulting determinant

$$\det \begin{bmatrix} \alpha_1' \\ \vdots \\ \alpha_i' \\ \vdots \\ \alpha_j' + \lambda\alpha_i' \\ \vdots \\ \alpha_n' \end{bmatrix} = \det \begin{bmatrix} \alpha_1' \\ \vdots \\ \alpha_i' \\ \vdots \\ \alpha_j' \\ \vdots \\ \alpha_n' \end{bmatrix} + \lambda \det \begin{bmatrix} \alpha_1' \\ \vdots \\ \alpha_i' \\ \vdots \\ \alpha_i' \\ \vdots \\ \alpha_n' \end{bmatrix} \begin{matrix} \\ \\ (i \\ \\ (j \\ \\ \\ \end{matrix} = \det A. \,!$$

THEOREM 10.9. A determinant can be expanded by any of its rows: that is,

$$\det A = \sum_{k=1}^{n} a_{ik} D_{ik} \qquad (i = 1, 2, \ldots, n) \tag{10.8}$$

where D_{ik} is the cofactor of a_{ik} (see (10.6)).

Proof. Interchanging the ith row $(i > 1)$ of $\det A$ one by one with the rows above it we can make it the first row while the order of the rest of the rows remains unchanged. In this procedure we are performing $(i - 1)$ interchanges of neighbouring rows, and thus by Theorem 10.6 the resulting determinant is the $(-1)^{i-1}$ multiple of $\det A$. Expanding this determinant by (10.4) we get

$$(-1)^{i-1} \det A = \sum_{k=1}^{n} a_{ik}(-1)^{1+k} \det A_{ik} .$$

Hence

$$\det A = \sum_{k=1}^{n} a_{ik}(-1)^{-i+1}(-1)^{1+k} \det A_{ik}$$

$$= \sum_{k=1}^{n} a_{ik}(-1)^{k-i} \det A_{ik} .$$

Since $(-1)^{k-i} = (-1)^{k+i}$, we get, taking (10.6) into account, the formula (10.8). !

THEOREM 10.10. If a row of a determinant is combined with the cofactors of another row, then the result is zero:

$$\sum_{k=1}^{n} a_{jk} D_{ik} = 0 \qquad (i, j = 1, 2, \ldots, n;\ i \neq j) .$$

Proof. Applying (10.8) we have

$$\det A + \sum_{k=1}^{n} a_{jk} D_{ik} = \sum_{k=1}^{n} a_{ik} D_{ik} + \sum_{k=1}^{n} a_{jk} D_{ik} = \sum_{k=1}^{n} (a_{ik} + a_{jk}) D_{ik}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} + a_{j1} & a_{i2} + a_{j2} & \ldots & a_{in} + a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \ldots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} .$$

However, the last determinant derives from $\det A$ by adding the jth row to the ith row $(i \neq j)$. Thus, by Theorem 10.8. the last determinant is equal to $\det A$:

$$\det A + \sum_{k=1}^{n} a_{jk} D_{ik} = \det A ,$$

that is,

$$\sum_{k=1}^{n} a_{jk} D_{ik} = 0 , \qquad i \neq j . !$$

The properties discussed up to now concern the rows of a determinant. We turn now to the columns.

Theorem 10.11. A determinant can be expanded in terms of its first column: that is,

$$\det A = \sum_{i=1}^{n} a_{i1} D_{i1},$$

where D_{i1} is the cofactor of a_{i1} (see (10.6)).

Proof (by mathematical induction for the order n of the determinant). By (10.5) the statement is obviously true for $n = 2$. Assume that it is valid for all $(n-1)$th-order determinants and consider the nth-order determinant $\det A$. If the latter is expanded in terms of its first row we have

$$\det A = a_{11} \det A_{11} + \sum_{k=2}^{n} a_{1k}(-1)^{1+k} \det A_{1k}.$$

$\det A_{1k}$ $(k = 2, 3, \ldots, n)$ is of order $(n-1)$ and by the assumption of induction it can be expanded in terms of its first column (the cofactor of a_{i1} in $\det A_{1k}$ is $(-1)^{(i-1)+1} \det A_{1k,i1}$, $(i = 2, 3, \ldots, n)$):

$$\begin{aligned} \det A &= a_{11} \det A_{11} + \sum_{k=2}^{n} a_{1k}(-1)^{1+k} \sum_{i=2}^{n} a_{i1}(-1)^{i-1+1} \det A_{1k,i1} \\ &= a_{11} \det A_{11} + \sum_{k=2}^{n} \sum_{i=2}^{n} a_{1k} a_{i1}(-1)^{i+k+1} \det A_{1k,i1}. \qquad (10.9) \end{aligned}$$

Now, expand formally $\det A$ in terms of its first column, i.e. consider the combination

$$\begin{aligned} \sum_{i=1}^{n} a_{i1} D_{i1} &= \sum_{i=1}^{n} a_{i1}(-1)^{i+1} \det A_{i1} \\ &= a_{11} \det A_{11} + \sum_{i=2}^{n} a_{i1}(-1)^{i+1} \det A_{i1}. \end{aligned}$$

We expand $\det A_{i1}$ in terms of its first row $(i = 2 \ldots, n)$. We get (taking into account that the cofactor of a_{1k} in $\det A_{i1}$ is $(-1)^{1+(k-1)}$ det $A_{i1,1k}$ $(k = 2, 3, \ldots, n)$)

$$\sum_{i=1}^{n} a_{i1} D_{i1} = a_{11} \det A_{11} + \sum_{i=2}^{n} a_{i1}(-1)^{i+1} \sum_{k=2}^{n} a_{1k}(-1)^{1+(k-1)} \det A_{i1,1k}.$$

Since $A_{i1,1k} = A_{1k,i1}$, we get from the last expression

$$\sum_{i=1}^{n} a_{i1} D_{i1} = a_{11} \det A_{11} + \sum_{i=2}^{n} \sum_{k=2}^{n} a_{i1} a_{1k}(-1)^{i+k+1} \det A_{1k,i1} = \det A$$

by (10.9). !

According to the previous theorem the first column plays a role analogous to the role of the first row in the defining expansion (10.7).

This means that using the expansion in terms of the first column, established in Theorem 10.11, we may proceed step by step in accordance with the construction for the rows and prove analogous theorems for the columns. That is to say, we have

COROLLARY 10.12. Theorems 10.1–10.10 remain valid if everywhere in them the word row is replaced by the word column.

In particular, we have

COROLLARY 10.13. A determinant can be expanded by any of its columns:

$$\det A = \sum_{i=1}^{n} a_{ik} D_{ik}, \qquad (k = 1, 2, \ldots, n), \tag{10.10}$$

where D_{ik} is the cofactor of a_{ik} (see (10.6)).

COROLLARY 10.14. The value of a determinant does not change if its rows are interchanged with its columns:

$$\det A = \det A' \tag{10.11}$$

(see Definition 8.3 and the remarks following it).

Proof. The statement is self-evident since the expansion of $\det A$, say, in terms of its first row yields terms identical to the terms in the expansion of $\det A'$ in terms of the first column of the latter. !

We prove one more important identity concerning the product of determinants of the same order. Some new notation and a lemma are needed.

Consider a matrix $A = [a_{ik}]$ of type $m \times n$, a matrix $B = [b_{jl}]$ of type $(M - m) \times (N - n)$, a matrix $C = [c_{jk}]$ of type $(M - m) \times n$, and a matrix $D = [d_{il}]$ of type $m \times (N - n)$, where $M \geq m$, $N \geq n$. The $M \times N$ matrix $U = [u_{ik}]$ whose elements are

$$\begin{aligned}
u_{ik} &= a_{ik}, && (i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n),\\
u_{i,n+l} &= d_{il}, && (i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, N - n),\\
u_{m+j,k} &= c_{jk}, && (j = 1, 2, \ldots, M - m;\ k = 1, 2, \ldots, n),\\
u_{m+j,n+k} &= b_{jl}, && (j = 1, 2, \ldots, M - m;\ l = 1, 2, \ldots, N - n),
\end{aligned}$$

is denoted by

$$U = \begin{bmatrix} A & D \\ C & B \end{bmatrix}, \tag{10.12}$$

and U is said to be *partitioned* into the submatrices A, B, C, and D, the latter being called the *blocks* of the partitioning. (A matrix, of course, can be partitioned in several ways into blocks and more than four blocks may be created; however, (10.12) is enough for our purpose.

Note that (10.2) is also a partitioning of the matrix into column vectors and row vectors, respectively.)

LEMMA 10.15. Consider

the square matrix $A = [a_{ik}]$ of order n,

the square matrix $B = [b_{jl}]$ of order p,

the matrix $C = [c_{jk}]$ of type $p \times n$

and the zero matrix O of type $n \times p$ and set

$$U = \begin{bmatrix} A & O \\ C & B \end{bmatrix};$$

U is obviously a square matrix of order $(n + p)$ and

$$\det U = \det A \cdot \det B.$$

Proof (by mathematical induction for the order n of the matrix A). If $n = 1$, then

$$\det U = \begin{vmatrix} a_{11} & 0 & 0 & \dots & 0 \\ c_{11} & b_{11} & b_{12} & \dots & b_{1p} \\ c_{21} & b_{21} & b_{22} & \dots & b_{2p} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ c_{p1} & b_{p1} & b_{p2} & \dots & b_{pp} \end{vmatrix} = a_{11} \det B = \det A \cdot \det B .$$

Assume that our statement is valid if the upper left block in the partitioning of U is of order $(n - 1)$ (for arbitrary p). Let A be of order n; then, expanding U in terms of its first row and taking into account that $u_{1,n+l} = 0$ $(l = 1, 2, \dots, p)$,

$$\det U = \sum_{k=1}^{n} a_{1k}(-1)^{1+k} \det U_{1k} ,$$

where U_{1k} is the submatrix of U corresponding to u_{1k}. But by the assumption of induction,

$$\det U_{1k} = \begin{vmatrix} a_{21} & \dots & a_{2,k-1} & a_{2,k+1} & \dots & a_{2n} & 0 & \dots & 0 \\ a_{31} & \dots & a_{3,k-1} & a_{3,k+1} & \dots & a_{3n} & 0 & \dots & 0 \\ \cdot & \dots & \cdot & \cdot & \dots & \cdot & \cdot & \dots & \cdot \\ a_{n1} & \dots & a_{n,k-1} & a_{n,k+1} & \dots & a_{nn} & 0 & \dots & 0 \\ c_{11} & \dots & c_{1,k-1} & c_{1,k+1} & \dots & c_{1n} & b_{11} & \dots & b_{1p} \\ \cdot & \dots & \cdot & \cdot & \dots & \cdot & \cdot & \dots & \cdot \\ c_{p1} & \dots & c_{p,k-1} & c_{p,k+1} & \dots & c_{pn} & b_{p1} & \dots & b_{pp} \end{vmatrix} = \det A_{1k} \cdot \det B.$$

Substituting this into the last expression for det U we get

$$\det U = \sum_{k=1}^{n} a_{1k}(-1)^{1+k} \det A_{1k} \cdot \det B = \det A \cdot \det B\,.!$$

THEOREM 10.16. The determinant of the product of the matrices $A = [a_{ik}]$ and $B = [b_{ik}]$ of order n is equal to the product of their determinants:

$$\det(AB) = \det A \cdot \det B.$$

Proof. The proof is first given for the case $n = 2$ to show clearly what is happening in the general case. Consider

$$\det U = \begin{vmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ -1 & 0 & b_{11} & b_{12} \\ 0 & -1 & b_{21} & b_{22} \end{vmatrix}.$$

On the one hand, by Lemma 10.15 det U = det A · det B. On the other hand, add the a_{11} multiple of the third row and the a_{12} multiple of the fourth row to the first row and then add the a_{21} multiple of the third row and the a_{22} multiple of the fourth row to the second row. By Theorem 10.8 the value of det U does not change, but the expansion in terms of the first column becomes easy:

$$\det U = \begin{vmatrix} 0 & 0 & a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ 0 & 0 & a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \\ -1 & 0 & b_{11} & b_{12} \\ 0 & -1 & b_{21} & b_{22} \end{vmatrix}$$

$$= (-1)(-1)\begin{vmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{vmatrix} = \det(AB)\,.$$

Thus for $n = 2$ the statement has been proved. The proof in the general case is similar. If the matrices A and B of order n are given, we consider the matrix

$$U = \begin{vmatrix} A & O \\ -I & B \end{vmatrix}$$

where O and I are the zero matrix and the unit matrix respectively, of order n. On the one hand, by Lemma 10.15 we have det U = det A · det B. On the other hand, add the a_{ik} multiple of the $(n + k)$th row $(k = 1, 2, \ldots, n)$ to the ith row $(i = 1, 2, \ldots, n)$. Applying Theorem 10.8, we have

$$\det U = \begin{vmatrix} O & C \\ -I & B \end{vmatrix},$$

where the elements of the matrix $C = [c_{ik}]$ are

$$c_{ik} = \sum_{l=1}^{n} a_{il} b_{lk}, \qquad (i = 1, 2, \ldots, n),$$

i.e. $C = AB$. Expanding the last expression of $\det U$ in terms of its first column and the minors occurring in the expansion in terms of their respective first columns, etc., we get after n steps

$$\det U = [(-1)^{n+2}(-1)]^n \det C = (-1)^{n(n+3)} \det C = \det (AB). \,!$$

In conclusion, we show some applications of the properties of determinants.

EXAMPLE 10.1. The following determinant

$$V_n(x_1 x_2, \ldots, x_n) = \begin{vmatrix} 1 & x_1 & x_1^2 \ldots x_1^{n-1} \\ 1 & x_2 & x_2^2 \ldots x_2^{n-1} \\ \ldots & \ldots & \ldots \\ 1 & x_n & x_n^2 \ldots x_n^{n-1} \end{vmatrix}$$

is called *Vandermonde's determinant* of order n ($x_1, x_2, \ldots, x_n$ are arbitrary real or complex numbers and are called the *generating elements* of Vandermonde's determinant). Before expanding Vandermonde's determinant we add the $(-x_1)$ multiple of the $(n-1)$th column to the last column, the $(-x_1)$ multiple of the $(n-2)$th column to the $(n-1)$th column etc., and the $(-x_1)$ multiple of the first column to the second column (by Theorem 10.8, applied to the columns, the value of the determinant does not change). Then the determinant is expanded in terms of its first row while the common factors are taken out from the respective rows (applying Theorem 10.1):

$$V(x_1, \ldots, x_n) = \begin{vmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & x_2 - x_1 & x_2^2 - x_1 x_2 & \ldots & x_2^{n-1} - x_1 x_2^{n-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & x_n - x_1 & x_n^2 - x_1 x_n & \ldots & x_n^{n-1} - x_1 x_n^{n-2} \end{vmatrix}$$

$$= (x_2 - x_1)(x_3 - x_1) \ldots (x_n - x_1) \begin{vmatrix} 1 & x_2 & x_2^2 \ldots x_2^{n-2} \\ 1 & x_3 & x_3^2 \ldots x_3^{n-2} \\ \cdot & \cdot & \cdot \;\cdots\; \cdot \\ 1 & x_n & x_n^2 \ldots x_n^{n-2} \end{vmatrix}$$

$$= (x_2 - x_1)(x_3 - x_1) \ldots (x_n - x_1) V_{n-1}(x_2, \ldots, x_n).$$

Thus the expansion of $V_n(x_1, \ldots, x_n)$ has been reduced to the expansion of the $(n-1)$th-order determinant $V_{n-1}(x_2, \ldots, x_n)$. Treating the latter similarly we get

$$V_{n-1}(x_2, \ldots, x_n) = (x_3 - x_2)(x_4 - x_2) \ldots (x_n - x_2) V_{n-2}(x_3, \ldots, x_n).$$

Proceeding further, after $(n - 1)$ steps we get

$$V_n(x_1, \ldots, x_n) = \prod_{i>k} (x_i - x_k) . \tag{10.13}$$

Expression (10.13) shows that a Vandermonde's determinant is zero if and only if it has at least two equal generating elements.

Example 10.2. Assume that an (x_1, x_2, x_3) Cartesian orthogonal coordinate system has been introduced in ordinary space (see Section 4) based on the orthonormal base $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ (see Definition 3.2) with origin at the point O. Three points A, B, C, say, in the x_1x_2 plane define a triangle. If the coordinates of these points are $A(a_1, a_2)$, $B(b_1, b_2)$ and $C(c_1, c_2)$ then the *area S of the triangle ABC* is

$$S = \pm \frac{1}{2} \begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix} . \tag{10.14}$$

Formula (10.14) can be proved as follows. According to the remark following Definition 2.3 we have for the area

$$S = \frac{1}{2} \left| \overrightarrow{AB} \times \overrightarrow{AC} \right| .$$

But applying (3.15)

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ b_1 - a_1 & b_2 - a_2 & 0 \\ c_1 - a_1 & c_2 - a_2 & 0 \end{vmatrix} = \mathbf{e}_3 \begin{vmatrix} b_1 - a_1 & b_2 - a_2 \\ c_1 - a_1 & c_2 - a_2 \end{vmatrix}$$

$$= \mathbf{e}_3 \begin{vmatrix} a_1 & a_2 & 1 \\ b_1 - a_1 & b_2 - a_2 & 0 \\ c_1 - a_1 & c_2 - a_2 & 0 \end{vmatrix} ,$$

where the validity of the last step can be checked by expanding the last determinant by its last column. Thus

$$S = \pm \frac{1}{2} \begin{vmatrix} a_1 & a_2 & 1 \\ b_1 - a_1 & b_2 - a_2 & 0 \\ c_1 - a_1 & c_2 - a_2 & 0 \end{vmatrix} .$$

Adding the first row to the second and to the third according to Theorem 10.8 we get (10.14). !

Example 10.3. We show that the volume V of the tetrahedron whose vertices are given in a Cartesian orthogonal coordinate system by $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$, $C(c_1, c_2, c_3)$, and $D(d_1, d_2, d_3)$ is

$$V = \pm\frac{1}{6}\begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{vmatrix}. \tag{10.15}$$

Obviously, the volume of the tetrahedron is the 1/6 multiple of the volume of the parallelepiped spanned by the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$, i.e., according to the remarks following (2.12),

$$V = \pm\frac{1}{6}(\overrightarrow{AB}\,\overrightarrow{AC}\,\overrightarrow{AD}).$$

(See (2.12).) Applying formula (3.16) we have

$$V = \pm\frac{1}{6}\begin{vmatrix} b_1 - a_1 & b_2 - a_2 & b_3 - a_3 \\ c_1 - a_1 & c_2 - a_2 & c_3 - a_3 \\ d_1 - a_1 & d_2 - a_2 & d_3 - a_3 \end{vmatrix}$$

$$= \pm\frac{1}{6}\begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 - a_1 & b_2 - a_2 & b_3 - a_3 & 0 \\ c_1 - a_1 & c_2 - a_2 & c_3 - a_3 & 0 \\ d_1 - a_1 & d_2 - a_2 & d_3 - a_3 & 0 \end{vmatrix}$$

where the validity of the last step can be checked by expanding the last determinant in terms of its last column. Adding the first row to the second, to the third, and to the fourth according to Theorem 10.8 we get (10.15). !

11. The Inverse of a Matrix

Elementary operations on matrices were discussed in Section 8. However, division by a matrix has not been defined. While in the set of the real or complex numbers division can be performed by any non-zero number, this is not so in the set of the matrices. We repeat here what was said at the end of Section 2. Division (in the set of the real or complex numbers) is based on the fact that if $a \neq 0$ then $ax = b$ has one and only one solution, $x = b/a$; in particular, if $b = 0$ then $x = 0$. This is not so with matrices. The product of two matrices may be the zero matrix even if none of the factors is the zero matrix. For example:

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

However, division by certain square matrices, the reciprocal or inverse of some square matrices, can be defined.

We recall that if a is a real or complex number then the reciprocal of a is the number $a^{-1} = 1/a$ for which $aa^{-1} = 1$. It is well known that each non-zero number has a uniquely determined reciprocal. The inverse of a square matrix is defined analogously.

DEFINITION 11.1. Let A be a square matrix of order n; the nth-order matrix denoted by A^{-1} is called the *inverse of A* if

$$AA^{-1} = A^{-1}A = I$$

where I is the unit matrix of order n.

The following expressions are used extensively: a square matrix is called *regular* if its determinant is non-zero; a square matrix is called *singular* if its determinant is zero.

First we show that not every matrix different from the zero matrix has an inverse. More precisely, *if A is singular then it has no inverse.* Let us assume, contrary to our statement, that $\det A = 0$ and still there is a matrix A^{-1} for which, for instance, the equation $AA^{-1} = I$ holds. If the determinants of both sides of the last equality are taken and Theorem 10.16 is applied to the left-hand side we get $\det A \cdot \det A^{-1} = 1$, i.e. $0 \cdot \det A^{-1} = 1$, or $0 = 1$, a contradiction. This proves our statement.

On the other hand, after some preparation we can show that every regular matrix has a uniquely determined inverse.

DEFINITION 11.2. Let us denote by D_{ik} the cofactor corresponding to the element a_{ik} of the nth-order matrix A, $(i, k = 1, 2, \ldots, n$, see (10.6)); the square matrix $C = [c_{ik}]$ of order n whose elements are

$$c_{ik} = D_{ki} \quad (i,\ k = 1, 2, \ldots, n)$$

is called the *associate matrix* of A.

In other words, the associate matrix of A is obtained by replacing each element with its cofactor and then transposing the matrix. Obviously each square matrix has one and only one associate. We note that the associate is often called the adjoint of A; but the name adjoint is already used in a different context.

LEMMA 11.1. If A is a matrix of order n and C is its associate then their product is a diagonal matrix whose elements in the main diagonal are all equal to $\det A$:

$$AC = CA = \begin{bmatrix} \det A & 0 \ldots & 0 \\ 0 & \det A \ldots & 0 \\ . & \ldots. & . \\ 0 & 0 \ldots & \det A \end{bmatrix}.$$

Proof. Introducing the notation $AC = [u_{ik}]$, $CA = [v_{ik}]$ for the product matrices, we have

$$u_{ik} = \sum_{l=1}^{n} a_{il} c_{lk} = \sum_{l=1}^{n} a_{il} D_{kl} = \delta_{ik} \det A$$

and

$$v_{ik} = \sum_{l=1}^{n} c_{il} a_{lk} = \sum_{l=1}^{n} D_{li} a_{lk} = \delta_{ik} \det A$$

where Definition 11.2, Theorems 10.9, 10.10, and Corollary 10.12 have been applied. !

THEOREM 11.2. If A is a regular square matrix then it has one and only one inverse A^{-1}, namely

$$A^{-1} = \frac{1}{\det A} C, \tag{11.1}$$

where C is the associate of A.

Proof. Since A is regular, $\det A \neq 0$ and the matrix on the right-hand side of (11.1) exists. This matrix is the inverse of A, since by our lemma

$$A \frac{1}{\det A} C = \frac{1}{\det A} CA$$

$$= \frac{1}{\det A} \begin{bmatrix} \det A & 0 \dots & 0 \\ 0 & \det A \dots & 0 \\ . & \dots. & . \\ 0 & 0 \dots & \det A \end{bmatrix} = I .$$

We have to prove that there is no inverse but (11.1). Assume that for some matrix X we have

$$AX = I \text{ or } XA = I.$$

Multiply the first equation by (11.1) (which is already rightly denoted by A^{-1}) from the left, and the second equation by (11.1) from the right. We get

$$A^{-1}(AX) = A^{-1}I \text{ or } (XA)A^{-1} = IA^{-1}.$$

Applying the associative law and (8.16) we have

$$(A^{-1}A)X = A^{-1} \text{ or } X(AA^{-1}) = A^{-1}.$$

Since A^{-1} is an inverse of A, the products in the brackets are equal to the unit matrix. Applying (8.16) again we get finally from both equations

$$X = A^{-1}. \; !$$

We prove now some simple theorems concerning the inverse of a matrix.

Theorem 11.3. If A is a regular matrix then

$$\det A^{-1} = \frac{1}{\det A}.$$

Proof. Take the determinant of both sides of the equality $AA^{-1} = I$. Applying Theorem 10.16 we get

$$\det A \cdot \det A^{-1} = 1. \;!$$

Theorem 11.4. If A is a regular matrix then its inverse matrix is also regular and

$$(A^{-1})^{-1} = A.$$

Proof. This follows at once from the previous theorem and Definition 11.1. !

Theorem 11.5. If A is a regular matrix then its transpose A' is also regular and

$$(A')^{-1} = (A^{-1})'. \tag{11.2}$$

Proof. The first part of the theorem follows from (10.11). In proving 11.2) we apply formula (8.23):

$$A'(A^{-1})' = (A^{-1}A)' = I' = I. \,!$$

Theorem 11.6. If A and B are regular matrices of the same order then AB is also regular and

$$(AB)^{-1} = B^{-1}A^{-1}. \tag{11.3}$$

Proof. The first part of the theorem follows from Theorem 10.16. In proving (11.3) we apply the associative law and the definition of the inverse

$$(AB)\,(B^{-1}A^{-1}) = A(BB^{-1})\,A^{-1} = AIA^{-1} = AA^{-1} = I. \,!$$

12. The Rank of a Matrix

This Section deals with the important problem of how to determine the number of linearly independent row or column vectors in a given matrix. The results will be used in the discussion of systems of linear equations.

Consider the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \dots & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \tag{12.1}$$

of type $m \times n$. Its column vectors and row vectors will be denoted by $a_1, \ldots, a_n$ and $\alpha'_1, \ldots, \alpha'_m$, respectively, and notations similar to (10.2) will be used for A.

Definition 12.1. If the row vectors $\alpha'_1, \ldots, \alpha'_m$ of the matrix (12.1) span an r-dimensional subspace $\mathcal{H}^r$ in the linear space $\mathcal{S}^n$ of the n-dimensional row vectors (cf. Definition 9.5), then we say that the *rank of the matrix* A is r and denote this by $r(A) = r$; the subspace $\mathcal{H}^r$ will be called the *row space* of A.

Definition 12.1′. If the maximal number of linearly independent row vectors in the matrix (12.1) is r then we say that the *rank* of A is r and denote this by $r(A) = r$.

The two alternative definitions given above for the rank of a matrix are equivalent. This follows from Theorem 9.12 and Corollary 9.13 (applied to the space $\mathcal{S}^n$ of the row vectors).

It is clear from Definition 12.1′ that

$$0 \leq r(A) \leq \min(m, n). \tag{12.2}$$

It is also clear that $r(A) = 0$ if and only if A is the zero matrix. We now show that if certain noperatios, called *elementary row operations* are performed on the matrix A then (though we get matrices different from A) the rank does not change.

Theorem 12.1. The rank of the matrix A does not change if
a) two rows are interchanged or
b) a row is multiplied by a non-zero number or
c) an arbitrary scalar multiple of a row is added to *another* row.

Proof. It is clear that operations a) and b) do not alter the dimension of the subspace spanned by the row vectors of the matrix. We have still to prove that operation c) does not alter the rank either. Let λ be an arbitrary number, add the λ multiple of the ith row to the jth row $(i \neq j)$, and denote the resulting matrix by B:

$$B = \begin{vmatrix} \alpha'_1 \\ \vdots \\ \alpha'_i \\ \vdots \\ \alpha'_j + \lambda\alpha'_i \\ \vdots \\ \alpha'_m \end{vmatrix}.$$

Since the row vectors of B are linear combinations of the row vectors of A, therefore by Corollary 9.14 (applied to $\mathcal{S}^n$) the number of linearly independent row vectors among them is at most $r(A)$, i.e. $r(B) \leq r(A)$.

However, we may get A from B by adding the $(-\lambda)$ multiple of the ith row vector (of B) to the jth row vector (of B). Thus interchanging the role of A and B in the preceding part of the proof shows that $r(A) \leq r(B)$; hence $r(A) = r(B)$. !

Definition 12.2. If the matrices A and B are of the same type and have the same rank, we say that they are *equivalent* and denote this by $A \sim B$.

By Theorem 12.1 matrices derived from each other by elementary row operations a), b), c) are equivalent.

Now we approach the point where we can give the main reason why determinants have been introduced (cf. the beginning of Section 10). If a matrix A of arbitrary type is given we can talk about its *submatrices* in a similar sense to Definition 10.1 (for square matrices), i.e. by deleting some rows and columns of A we get a submatrix. There is a strong connection between the nullity of the determinants of certain square submatrices on the one hand and the number of linearly independent row vectors on the other.

Theorem 12.2. If the rank of the $m \times n$ matrix A is r then A has a regular square submatrix of order r and all submatrices of order higher than r are singular.

Proof. We show first that all submatrices of order $(r+1)$ are singular. Consider an arbitrary submatrix of order $(r+1)$. Since the rank of A is r, therefore the $(r+1)$ rows occurring in this submatrix are linearly dependent. Thus at least one of them can be expressed as a linear combination of the rest. Applying Theorem 10.2 the determinant of this submatrix is a linear combination of determinants having two equal rows each. But by Theorem 10.7 all these determinants are zero. We may consider, for instance, the submatrix

$$M = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1,r+1} \\ \cdot & \cdot & \dots & \cdot \\ a_{r1} & a_{r2} & \dots & a_{r,r+1} \\ a_{r+1,1} & a_{r+1,2} & \dots & a_{r+1,r+1} \end{vmatrix}$$

and may assume that, say, $\alpha'_{r+1} = \sum_{i=1}^{r} \lambda_i \alpha'_i$. Hence

$$\det M = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1,r+1} \\ \cdot & \cdot & \dots & \cdot \\ a_{r1} & a_{r2} & \dots & a_{r,r+1} \\ \sum_{i=1}^{r} \lambda_i a_{i1} & \sum_{i=1}^{r} \lambda_i a_{i2} & \dots & \sum_{i=1}^{r} \lambda_i a_{i,r+1} \end{vmatrix}$$

$$= \sum_{i=1}^{r} \lambda_i \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1,r+1} \\ \cdot & \cdot & \dots & \cdot \\ a_{r1} & a_{r2} & \dots & a_{r,r+1} \\ a_{i1} & a_{i2} & \dots & a_{i,r+1} \end{vmatrix} = 0.$$

Thus all submatrices of order $(r + 1)$ are zero and consequently all submatrices of order higher than $(r + 1)$ are also zero.

Now we show that a regular submatrix of order r exists. Assume, contrary to our statement, that the highest order regular submatrix is of order s where $s < r$. We may assume without loss of generality that

$$d = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1s} \\ a_{21} & a_{22} & \dots & a_{2s} \\ \cdot & \cdot & \dots & \cdot \\ a_{s1} & a_{s2} & \dots & a_{ss} \end{vmatrix} \neq 0\,.$$

If we attach to this determinant a row and a column in the following way we get a determinant that is zero:

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1s} & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2s} & a_{2k} \\ \cdot & \cdot & \dots & \cdot & \cdot \\ a_{s1} & a_{s2} & \dots & a_{ss} & a_{sk} \\ a_{j,1} & a_{j,2} & \dots & a_{j,s} & a_{j,k} \end{vmatrix} = 0$$

$$(k = 1, 2, \dots, n;\ j = s + 1, \dots, m),$$

since if $1 \leq k \leq s$, then the last determinant has two equal columns (see Theorem 10.7 and Corollary 10.12), and if $s < k \leq n$, then this is a minor corresponding to a submatrix of order $(s + 1)$. Expand the last determinant having the value zero in terms of its last column:

$$\sum_{i=1}^{s} a_{ik}\lambda_i + a_{jk}\lambda_j = 0, \qquad k = 1, 2, \dots, n,$$

where the λ_i cofactors do not depend on k and $\lambda_j = \pm d \neq 0$. If we take into account that a_{ik} is the kth coordinate of the ith row vector α_i', the last equality can be written

$$\sum_{i=1}^{s} \lambda_i \alpha_i' + \lambda_j \alpha_j' = 0, \quad \lambda_j \neq 0.$$

Hence any row vector is a linear combination of the row vectors

$\alpha_1', \alpha_2', \ldots, \alpha_s'$ and, as a consequence (see Corollary 9.14 applied to $\mathcal{S}^n$) the rank is s at most. This is a contradiction. !

COROLLARY 12.3. If the maximal regular submatrix of the matrix A is of order r, then $r(A) = r$.

Thus an equivalent definition of the rank of a matrix would be: the order of the regular submatrix of maximal order.

LEMMA 12.4. The rank of the transpose A' of a matrix A is equal to the rank of A:

$$r(A') = r(A).$$

Proof. If the transpose of a matrix is formed then all its square submatrices are also transposed and by (10.11) the values of their determinants do not alter. !

THEOREM 12.5. If the column vectors $a_1, a_2, \ldots, a_n$ of the matrix (12.1) span an r-dimensional subspace $\mathcal{G}^r$ in the linear space $\mathcal{Q}^m$ of the m-dimensional column vectors, then $r(A) = r$.

Proof. The column vectors of A are the row vectors of its transpose A'; thus the theorem follows from the previous lemma. !

The subspace $\mathcal{G}^r$ in Theorem 12.5 is called the *column space* of A. By this theorem a definition of rank equivalent to Definition 12.1 would be: the dimension of the column space. Obviously the rank is equal also to the maximal number of linearly independent column vectors. This fact can be expressed also by saying that in any matrix *the maximal number of linearly independent row vectors is equal to the maximal number of linearly independent column vectors.*

The following theorem is analogous to Theorem 12.1.

THEOREM 12.6. The rank of a matrix does not change if the following *elementary column operations* are performed on it:

a) two columns are interchanged or

b) a column is multiplied by a non-zero number or

c) an arbitrary scalar multiple of a column is added to *another* column.

Proof. Analogous to the proof of Theorem 12.1. !

By the last theorem matrices obtained from each other by elementary column operations are equivalent.

The elementary row and column operations yield a practical way to determine the rank of a matrix. It is easy to see that *by elementary row and column operations one can get from any matrix an equivalent matrix such that in each row and in each column of the latter there exists at most one element different from zero. The rank of such a matrix is equal to the number of non-zero elements.* We now give an example.

Example 12.1. The rank of the matrix

$$A = \begin{bmatrix} 2 & -3 & 0 & 4 \\ 1 & -1 & 5 & 2 \\ 5 & -7 & 5 & 10 \end{bmatrix}$$

will be determined.

$$A \sim \begin{bmatrix} 2 & -3 & 0 & 2 \\ 1 & -1 & 1 & 1 \\ 5 & -7 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 4 & -6 & 1 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where the following elementary row and column operations have been performed:

first step: the third and the fourth columns have been multiplied by 1/5 and 1/2, respectively;
second step: the (-1) multiple, the 1 multiple, and the (-1) multiple of the third column have been added to the first, the second, and the fourth columns, respectively;
third step: the first, the second, and the fourth columns have been multiplied by 1/2, $(-1/3)$, and 1/2, respectively;
fourth step: the (-2) multiple of the first row has been added to the third row;
fifth step: the (-1) multiple of the first column has been added to the second and the fourth columns;
sixth step: the (-1) multiple of the second row has been added to the third row.

The rank of the last matrix is, clearly, 2 and hence also $r(A) = 2$.

In accordance with our remark at the beginning of Section 10, from Theorem 12.2 we get in the special case $m = n$:

Corollary 12.7. The row vectors (column vectors) of an nth-order square matrix A are linearly independent, or in other words they span the n-dimensional space $\mathbb{S}^n(\mathbb{Q}^n)$ if and only if A is regular, that is to say $\det A \neq 0$.

This Corollary is at the same time a generalization of Theorem 2.5, which was expressed in a form suitable to generalization after formula (3.16).

In conclusion we prove one more theorem for later use.

THEOREM 12.8. If $A = [a_{ik}]$ and $B = [b_{ik}]$ are square matrices of order n and there exists a regular matrix $T = [t_{ik}]$ of order n such that $B = TA$ or $B = AT$, then $A \sim B$, i.e. $r(A) = r(B)$.

Proof. We give the proof for the case $B = TA$. The other case can be treated similarly. The row vectors of A are denoted by α'_i, as before, and those of B by $\beta'_1, \ldots, \beta'_n$. We have

$$b_{ik} = \sum_{l=1}^{n} t_{il}\, a_{lk}, \quad (i, k = 1, 2, \ldots, n).$$

If in the last equation i is considered to be fixed and the values from 1 to n are given to k, it follows that

$$\beta'_i = \sum_{l=1}^{n} t_{il}\, \alpha'_l \qquad (i = 1, 2, \ldots, n).$$

According to the last equality the row vectors β'_i are elements of the row space of A. Hence the maximal number of linearly independent row vectors among them is not greater than $r(A)$ (see Corollary 9.14 applied to $\mathfrak{S}^n$). Thus $r(B) \leq r(A)$. But since T is regular, it has an inverse T^{-1}. Multiplying the equation $B = TA$ from the left by T^{-1} we get $A = T^{-1}B$. The previous reasoning can now be applied with an interchange of the roles of A and B. Hence we get $r(A) \leq r(B)$. Thus $r(A) = r(B)$. !

Exercises

III.1 $A = \begin{bmatrix} 3 & 2 \\ -1 & 5 \\ -4 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$;

solve the matrix equation

$$A + X = B.$$

III.2 $A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$;

verify the associativity by computing both sides of the identity

$$(AB)C = A(BC).$$

III.3 $A = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$;

a) show that A and B are not commutable;
b) determine the matrix

$$AB + BC + BA$$

using the distributive law.

III.4 A and B are square matrices of the same type. Prove or disprove the identity $A^2 - B^2 = (A + B)(A - B)$.
Check your result in the case when

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 2 \\ 0 & 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -2 & 4 \\ 3 & 5 & 1 \\ -1 & 6 & 0 \end{bmatrix}.$$

III.5 A and B are symmetric square matrices of the same type. Prove that the matrix $AB - BA$ has some zero elements.

III.6 Can you find a non-zero matrix B of type 3×1 such that

$$A \cdot B = O$$

if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$$

or if

$$A = \begin{bmatrix} 3 & -2 & 1 \\ -4 & 1 & -3 \\ -6 & 4 & -2 \end{bmatrix}?$$

III.7

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix};$$

a) show that $AB = A$ and $BA = B$;

b) use a) to prove that $A^2 = A$ and $B^2 = B$;

c) show that if $C = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$, then

$AC \neq A$, $CA \neq C$ but $C^2 = C$.

III.8 Prove that two symmetric matrices are commutable if and only if their product is symmetric.

III.9 Determine the matrices of type 2×2 for which

$$A^2 = I.$$

III.10 Determine all matrices of the form

$$A = \begin{bmatrix} z_1 & z_2 \\ z_2 & z_3 \end{bmatrix}$$

where the elements z_i $(i = 1, 2, 3)$ are complex numbers, for which $A^2 = I$.

III.11 z_1 and z_2 are complex numbers. Show that the product of two matrices of the form $\begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix}$ is again of the same form.

Are such matrices commutable?

III.12 $A = [a_1, a_2, \ldots, a_n]$ where a_k is the kth column vector of A; the matrix C differs from A in its last column, namely:

$$C = [a_1, a_2, \ldots, a_{n-1}, c_1a_1 + c_2a_2 + \ldots + c_{n-1}a_{n-1} + a_n],$$

where c_i, $i = 1, 2, \ldots, n - 1$ are complex numbers. Find a matrix B such that

$$AB = C.$$

III.13 Does a non-zero hermitian matrix A exist such that $A^2 = O$?

III.14 Do non-zero hermitian matrices A and B exist such that $AB = O$?

III.15 For any natural number n and real a find

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^n.$$

III.16 Prove that the column vectors

$$a_1, \ldots, ca_k, \ldots, a_n \quad (n \geq 2,\ c \neq 0 \text{ a real number})$$

are linearly independent if and only if the column vectors

$$a_1, \ldots, a_k, \ldots, a_n$$

are linearly independent.

III.17 Prove that the column vectors

$$a_1, a_2, \ldots, a_l, \ldots, a_{k-1}, a_k + ca_l, a_{k+1}, \ldots, a_n \quad (n \geq 2, c \neq 0 \text{ and } l \neq k)$$

are linearly independent if and only if the column vectors

$$a_1, a_2, \ldots a_l, \ldots a_k, \ldots a_n$$

are linearly independent.

III.18 Find the maximal number of linearly independent column vectors among

$$a_1 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix}.$$

III.19 Find a column vector a_3 such that

$$a_1 = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} \quad \text{and } a_3$$

should be linearly independent.

III.20 Find all the real numbers c for which the column vectors

$$a_1 = \begin{bmatrix} c \\ 1 \\ 1 \end{bmatrix}, \qquad a_2 = \begin{bmatrix} 1 \\ c \\ 1 \end{bmatrix}, \qquad a_3 = \begin{bmatrix} 1 \\ 1 \\ c \end{bmatrix}$$

are linearly dependent.

III.21 Determine the values of the following determinants

$$\text{a)} \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}, \qquad \text{b)} \begin{vmatrix} b & a & a \\ a & b & a \\ a & a & b \end{vmatrix},$$

where a, b, c are real numbers.

III.22 Show that the value of the determinant

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix}$$

is divisible by 8.

III.23 Solve the following equations for x:

$$\text{a)} \begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 2-x^2 & 2 & 3 \\ 2 & 3 & 1 & 5 \\ 2 & 3 & 1 & 7-x^2 \end{vmatrix} = 0;$$

$$\text{b)} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 7 \\ 1 & x & x^2 & x^3 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & x & x^2 & x^3 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ 1 & 2 & 4 & 8 \\ 1 & x & x^2 & x^3 \end{vmatrix} = 0.$$

III.24 Show that in the xy plane (i.e. in the planar orthogonal Cartesian coordinate system with horizontal axis x and vertical axis y) the equation of a line passing through the points (x_1, y_1) and (x_2, y_2) is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

III.25 Show that in the xy plane the equation of a circle passing through the non-collinear points (x_1, y_1), (x_2, y_2), (x_3, y_3) is

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

III.26 Show that in the xy plane the equation of the parabola passing through the non-collinear points (x_1, y_1), (x_2, y_2), (x_3, y_3) with different abscissae $(x_i \neq x_k,\ i \neq k)$ is

$$\begin{vmatrix} x^2 & x & 1 & -y \\ x_1^2 & x_1 & 1 & -y_1 \\ x_2^2 & x_2 & 1 & -y_2 \\ x_3^2 & x_3 & 1 & -y_3 \end{vmatrix} = 0.$$

III.27 Try to give in determinant form the equation of the plane in the xyz orthogonal Cartesian coordinate system passing through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3).

III.28 Find the value of parameter t for which the matrix

$$A = \begin{vmatrix} 1 & 0.3 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & t \end{vmatrix}$$

has an inverse.

III.29 $a_1, a_2, \ldots a_n$ are real numbers and $a_1 \neq 0$. Determine the inverse of the matrix

$$A = \begin{vmatrix} a_1 & 0 & 0 \ldots & 0 \\ a_2 & 1 & 0 \ldots & . \\ a_3 & 0 & 1 \ldots & . \\ \vdots & \vdots & \vdots & \vdots \\ a_{n-1} & 0 & 0 \ldots & 0 \\ a_n & 0 & 0 \ldots & 1 \end{vmatrix}.$$

III.30 Let $A^2 = -I$; what is the inverse of A?

III.31 A is a regular matrix, B a quadratic matrix of the same order, and $AB = O$; prove that $B = O$.

III.32 Find A^{-1} if

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

III.33

$$A = \begin{bmatrix} 1 & -8 & 9 \\ 2 & -1 & 3 \\ 1 & 2 & -1 \end{bmatrix};$$

can you find A^{-1}?

III.34 Find the rank of the matrix

$$B = \begin{bmatrix} 1 & -8 & 9 & -32 \\ 2 & -1 & 3 & -1 \\ 1 & 2 & -1 & 12 \end{bmatrix}.$$

III.35 Let $0 \leq a < 1$; prove that

$$A = \begin{bmatrix} a & a & 1 \\ a & 1 & a \\ 1 & a & a \end{bmatrix}$$

has an inverse.

III.36 Prove that

$$A = \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix}$$

is regular if and only if

$$a^2 + b^2 + c^2 + d^2 \neq 0.$$

III.37 A is a regular matrix; what is the inverse of the associate of A?

CHAPTER FOUR

SYSTEMS OF LINEAR EQUATIONS

13. Existence and Uniqueness of Solutions

Many mathematical problems lead, at the end, to a system of linear equations. In this Section we discuss the circumstances in which a system of linear equations has a solution and the conditions under which this solution is unique. But first we want to emphasize that whether or not a linear system has a solution does not depend on the relation of the number of unknowns to the number of equations. A linear system may have solutions even if it is 'overdetermined', i.e. the number of equations is greater than the number of unknowns, and a linear system may have no solution at all even if it is 'underdetermined', i.e. the number of equations is less than the number of unknowns. At the same time, if the number of equations is equal to the number of unknowns the system does not necessarily have a unique solution.

EXAMPLE 13.1. The linear system with two unknowns and three equations

$$\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 + 2x_2 &= 2 \\ 3x_1 + 3x_2 &= 3 \end{aligned}$$

has an infinite number of solutions: x_2 may be any number, and then $x_1 = 1 - x_2$.

EXAMPLE 13.2. The linear system with three unknowns and two equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + x_2 + x_3 &= 2 \end{aligned}$$

has no solution, since the same sum cannot be equal at the same time to 1 and to 2.

The general form of a system of linear equations is

$$\begin{array}{ccccccccc} a_{11}\, x_1 & + & a_{12}\, x_2 & + & \ldots & + & a_{1n}\, x_n & = & b_1 \\ a_{21}\, x_1 & + & a_{22}\, x_2 & + & \ldots & + & a_{2n}\, x_n & = & b_2 \\ \cdot & & \cdot & & \cdots & & \cdot & & \cdot \\ a_{m1} x_1 & + & a_{m2} x_2 & + & \ldots & + & a_{mn} x_n & = & b_m\, . \end{array} \tag{13.1}$$

Here the *coefficients* a_{ik} and the *constants* b_i $(i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n)$ are given real or complex numbers; $x_1, \ldots, x_n$ are the *unknowns*. An ordered n-tuple of numbers $(x_1, x_2, \ldots, x_n)$ is a *solution* if the substitution of this n-tuple into the system makes an identity of each equation. The matrix $A = [a_{ik}]$ of type $m \times n$ is the *coefficient matrix*. The m-dimensional column vector $b = \operatorname{col}[b_1, b_2, \ldots, b_m]$ is the *constant vector*. The n-dimensional column vector $x = \operatorname{col}[x_1, x_2, \ldots, x_n]$ is the *unknown vector*. With this notation, (13.1) can be put into the following equivalent matrix form

$$Ax = b\,. \tag{13.2}$$

The n-dimensional column vector x is a solution of (13.2) if and only if the ordered n-tuple corresponding to it is a solution of (13.1). Denote the columns of the coefficient matrix by $a_1, a_2, \ldots, a_n$, i.e. let

$$A = [a_1, a_2, \ldots, a_n].$$

Then (13.1) can be written in another equivalent form:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b. \tag{13.3}$$

Finally, denote the rows of the coefficient matrix by $\alpha_1', \alpha_2', \ldots, \alpha_m'$ i.e. let

$$A = \begin{bmatrix} \alpha_1' \\ \alpha_2' \\ \vdots \\ \alpha_m' \end{bmatrix}.$$

Then still another equivalent form of (13.1) is

$$\alpha_i' x = b_i, \quad (i = 1, 2, \ldots, m). \tag{13.4}$$

We use the forms (13.1), (13.2), (13.3) and (13.4) as alternatives, depending on which is the most suitable for the discussion.

As is clear from (13.3), to solve the system means to express b as a linear combination of the column vectors a_k $(k = 1, 2, \ldots, n)$. Denote the column space of the coefficient matrix A by $\mathcal{G}^r$ (see Theorem 12.5), i.e. $\mathcal{G}^r$ is the subspace spanned by $a_1, \ldots, a_n$ of the linear space $\mathcal{Q}^m$ of m-dimensional column vectors.

THEOREM 13.1. The linear system (13.3) has a solution if and only if $b \in \mathcal{G}^r$.

Proof. The statement follows readily from Corollary 9.13 and Theorem 9.11. !

This theorem can be put into another form if the matrix $C = [A, b]$ is introduced (see the remark following (10.12)). C is a matrix of type $m \times (n + 1)$ and is called the *augmented matrix* of the system.

THEOREM 13.2. The linear system (13.2) has a solution if and only if the rank of the coefficient matrix is equal to the rank of the augmented matrix:

$$r(C) = r(A)\,. \tag{13.5}$$

Proof. C has been formed from A by augmenting the latter with the column vector b. Hence, the number of linearly independent column vectors in C cannot be less than the number of linearly independent column vectors in A, i.e. $r(A) \leq r(C)$. If the system has a solution then $b \in \mathcal{G}^r$ and, consequently, the column space of C is the same as the column space of A, thus, $r(C) = r(A)$. Conversely, $r(C) = r(A)$ means that, augmenting the matrix A with the column vector b, the dimension of the column space does not increase. Hence $b \in \mathcal{G}^r$, and the linear system can be solved by Theorem **13.1**. !

We give an effective method of solution, yielding the solutions and checking condition (13.5) simultaneously, in the next Section. Let us turn now to the problem of uniqueness of the solution.

THEOREM 13.3. If in system (13.1) the common rank of the coefficient matrix and the augmented matrix is equal to the number of unknowns, i.e. if

$$r(A) = r(C) = n\,, \tag{13.6}$$

then the system has one and only one solution.

Note that condition (13.6) holds only if $m \geq n$, i.e. only if the number of equations is not less than the number of unknowns (since $r(A) \leq \min\,(m, n)$). In the next Section we shall see that condition $r(A) = n$ is also necessary for the system to have one solution at most.

Proof. If condition (13.6) holds then by the previous theorem the system has a solution. At the same time (13.6) means that the column space of A is of dimension n (therefore it will be denoted by $\mathcal{G}^n$), i.e. the column vectors $a_1, a_2, \ldots, a_n$ span an n-dimensional subspace and they form a base of $\mathcal{G}^n$. But then, by Theorem 9.11, there is not more than one linear combination of $a_1, \ldots, a_n$ equal to $b \in \mathcal{G}^n$. !

We have for the special case in which the number of unknowns is equal to the number of equations the following

THEOREM 13.4. If in system (13.1) $m = n$ and A is a regular matrix then (13.1) has one and only one solution

$$x = A^{-1}b. \tag{13.7}$$

Proof. By Corollary 12.7 $r(A) = n$. Since the augmented matrix C is of type $n \times (n+1)$, we have $r(A) \leq r(C) \leq n$. Hence $r(C) = n$ too. Thus by Theorem **13.3** there is one and only one solution. Substituting (13.7) into (13.2) we get

$$A(A^{-1}b) = (AA^{-1})b = Ib = b,$$

i.e. the only solution is given by (13.7). !

Note that if the expression (11.1) of the inverse matrix is substituted into (13.7) and the elements of the solution x are written out separately, we get

$$x_i = \frac{1}{\det A} \sum_{k=1}^{n} D_{ki} b_k, \qquad (i = 1, 2, \ldots, n) \tag{13.8}$$

where D_{ki} is the cofactor of a_{ki} in the matrix A. (13.8) is called *Cramer's rule*.

EXAMPLE 13.3. Let A be a regular matrix of order n and denote the column vectors of its inverse matrix by $a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1}$, i.e. let $A^{-1} = [a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1}]$. Let $e_k = [\delta_{ik}]$ $(k = 1, 2, \ldots, n)$ be the n-dimensional column vectors (δ_{ik} the Kronecker symbol) as in (9.9) and consider the linear system

$$Ax = e_k \qquad (k = 1, 2, \ldots, n). \tag{13.9}$$

By Theorem 13.4 the only solution of this system is

$$x_k = A^{-1} e_k \qquad (k = 1, 2, \ldots, n).$$

However, it is clear applying merely the rule of matrix multiplication, that

$$x_k = A^{-1} e_k = a_k^{-1} \quad (k = 1, 2, \ldots, n). \tag{13.10}$$

Thus, looking at (13.9) as n simultaneous systems of linear equations (each system having the same coefficient matrix) we see that the solutions of these simultaneous systems are the respective columns of the matrix A^{-1}.

14. Gauss's Method of Solution

Gauss's method for solving the linear system (13.1) consists of several steps that each produce a new linear system equivalent to the original, finally leading to a linear system from which the unknowns can be easily derived one by one. (Two linear systems are *equivalent* if the respective sets of solutions are the same.) This is the case, of course, when a solution exists. If the system has no solution, the method leads to contradiction. At each step only elementary row (and possibly column) operations (see Theorems 12.1 and 12.6) are performed on the augmented matrix of the system. The method yields itself to easy programming for a computer.

We may assume without loss of generality that $a_{11} \neq 0$ in (13.1). If not, we could insert as the first equation in (13.1) one in which the coefficient of x_1 is non-zero. (If we look at the coefficient matrix of the system, this operation means an interchange of two rows.) Leaving the first equation as it is we subtract the

$$\frac{a_{21}}{a_{11}}, \quad \frac{a_{31}}{a_{11}}, \quad \ldots, \quad \frac{a_{m1}}{a_{11}}$$

multiple of the first equation from the second, the third, and so on to the mth one, respectively. Thus we get the linear system

$$\begin{aligned} a_{11}^{(1)} x_1 + a_{12}^{(1)} x_2 + \ldots + a_{1n}^{(1)} x_n &= b_1^{(1)} \\ a_{22}^{(1)} x_2 + \ldots + a_{2n}^{(1)} x_n &= b_2^{(1)} \\ . \qquad \ldots \qquad . \quad & \; . \\ a_{m2}^{(1)} x_2 + \ldots + a_{mn}^{(1)} x_n &= b_m^{(1)} \end{aligned} \tag{14.1}$$

which is equivalent to (13.1). Incidentally, $a_{1k}^{(1)} = a_{1k}$, $(k = 1, 2, \ldots, n)$ and $b_1^{(1)} = b_1$ (new notation having been introduced for the sake of homogeneity). The coefficient matrix $A^{(1)}$ and the augmented matrix $C^{(1)}$ of system (14.1) is derived from A and C, respectively, by elementary row operations. Hence, by Theorem 12.1, $A^{(1)} \sim A$ and $C^{(1)} \sim C$ (see Definition 12.2).

We have $a_{11}^{(1)} = a_{11} \neq 0$, and we may assume without loss of generality that $a_{22}^{(1)} \neq 0$. Were this not the case either we could rearrange the system to have an equation in the second place in which the coefficient of x_2 is non-zero (interchanging two rows), or we could renumber the unknowns to have one in the second column with non-zero coefficient in the second equation. This is possible only if $a_{ik}^{(1)} = 0$ $(i = 2, 3, \ldots, m;\ k = 2, 3, \ldots, n)$, and in this case the method is terminated. Leaving the first and second equations as they are we subtract the

$$\frac{a_{32}^{(1)}}{a_{22}^{(1)}}, \quad \frac{a_{42}^{(1)}}{a_{22}^{(1)}}, \ldots, \frac{a_{m2}^{(1)}}{a_{22}^{(1)}}$$

multiple of the second equation from the third, the fourth, and so on to the mth one, respectively. Thus we get the linear system

$$\begin{aligned} a_{11}^{(2)} x_1 + a_{12}^{(2)} x_2 + a_{13}^{(2)} x_3 + \ldots + a_{1n}^{(2)} x_n &= b_1^{(2)} \\ a_{22}^{(2)} x_2 + a_{23}^{(2)} x_3 + \ldots + a_{2n}^{(2)} x_n &= b_2^{(2)} \\ a_{33}^{(2)} x_3 + \ldots + a_{3n}^{(2)} x_n &= b_3^{(2)} \\ . \qquad \ldots \qquad . \quad & \; . \\ a_{m3}^{(2)} x_3 + \ldots + a_{mn}^{(2)} x_n &= b_m^{(2)} \end{aligned}$$

which is equivalent to (14.1) and (13.1) and whose coefficient matrix $A^{(2)}$ and augmented matrix $C^{(2)}$ are obtained from $A^{(1)}$ and $C^{(1)}$, respectively, by elementary row (and possibly column) operations. Hence $A^{(2)} \sim A$ and $C^{(2)} \sim C$. $a_{11}^{(2)} = a_{11} \neq 0$, $a_{22}^{(2)} = a_{22}^{(1)} \neq 0$, and we may assume without loss of generality that $a_{33}^{(2)} \neq 0$. We may now perform the third step in the same way as the first and the second. The method terminates after a finite number of steps, say r, either because $r = m$ or because all coefficients in the $(r + 1)$th, $(r + 2)$th and so on to the mth equation are zero.

Thus we have reduced (13.1) to an equivalent linear system of the following form

$$\begin{array}{rcl}
a_{11}^{(r)} x_1 + a_{12}^{(r)} x_2 + \dots\dots\dots\dots + a_{1n}^{(r)} x_n &=& b_1^{(r)} \\
a_{22}^{(r)} x_2 + \dots\dots\dots\dots + a_{2n}^{(r)} x_n &=& b_2^{(r)} \\
\cdot \quad \dots\dots\dots\dots \quad \cdot && \cdot \\
a_{ii}^{(r)} x_i + \dots\dots\dots + a_{in}^{(r)} x_n &=& b_i^{(r)} \\
\dots\dots\dots \quad \cdot && \cdot \\
a_{rr}^{(r)} x_r + \dots + a_{rn}^{(r)} x_n &=& b_r^{(r)} \\
0 &=& b_{r+1}^{(r)} \\
&\vdots& \\
0 &=& b_m^{(r)} \,.
\end{array} \qquad (14.2)$$

(If $r = m$ then the $(r + 1)$th, etc. equations do not arise.) In (14.2)

$$a_{ii}^{(r)} \neq 0, \qquad (i = 1, 2, \dots, r)$$

and the coefficient matrix $A^{(r)}$ and the augmented matrix $C^{(r)}$ of system (14.2) are equivalent to A and C, respectively.

Obviously, all submatrices of order $(r + 1)$ of $A^{(r)}$ are singular (they contain at least one zero row) and

$$\begin{vmatrix} a_{11}^{(r)} & a_{12}^{(r)} & \dots & a_{1r}^{(r)} \\ 0 & a_{22}^{(r)} & \dots & a_{2r}^{(r)} \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & a_{rr}^{(r)} \end{vmatrix} = a_{11}^{(r)} a_{22}^{(r)} \dots a_{rr}^{(r)} \neq 0 \,.$$

Hence $r(A) = r(A^{(r)}) = r$. On the other hand it is easy to see that

$$r(C) = r(C^{(r)}) = r, \qquad \text{or} \qquad r + 1$$

depending on whether $b_{r+1}^{(r)} = b_{r+2}^{(r)} = \dots = b_m^{(r)} = 0$, or there exists $r + 1 \leq j \leq m$ such that $b_j^{(r)} \neq 0$. In the latter case the equation $0 = b_j^{(r)}$ is a contradiction and the system has no solution in accordance with Theorem 13.2 $\big(r(C) \neq r(A)\big)$. In the former case the last $m - r$ equations are identities and the system has solutions. (This is the case also when $r = m$ and the last $m - r$ equations do not occur at all.) We get the solutions by fixing arbitrarily the values of the unknowns $x_{r+1}, x_{r+2}, \dots, x_n$ and determining x_r from the rth equation (where the coefficient of x_r is non-zero.) Substituting these values of $x_{r+1}, \dots, x_n$, and x_r into the $(r - 1)$th equation we determine x_{r-1}, etc. and we solve the equations, proceeding, one by one, for x_{r-2}, $\dots, x_{r-3}, \dots, x_1$.

We can sum the method up as follows:

Theorem 14.1. The linear system (13.1) is equivalent to a system of the form (14.2) where $a_{ii}^{(r)} \neq 0$ $(i = 1, 2, \ldots, r)$ and $r = r(A)$; (13.1) has a solution if and only if $b_k^{(r)} = 0$ $(k = r + 1, r + 2, \ldots, m)$; for arbitrarily fixed values of $x_{r+1}, \ldots, x_n$ (14.2) determines the values of $x_r, x_{r-1}, \ldots, x_1$ in the solution uniquely.

Corollary 14.2. If for the linear system (13.1) $r(A) = r(C) < n$, then (13.1) has an infinite number of solutions (cf. Theorem 13.3).

As an application of Gauss's method, modified somewhat for the purpose, we now give an algorithm for the determination of the inverse of a regular square matrix.

In principle one can determine the inverse through the associate matrix by formula (11.1). However, this is highly impracticable because of the number of calculations involved when using determinants.

Let A be a regular matrix of order n and consider the n simultaneous systems (see Example 13.3)

$$Ax = e_k \quad (k = 1, 2, \ldots, n). \tag{14.3}$$

The coefficient matrix of each system is A, the augmented matrix of the kth system is $[A, e_k]$, and we may call the matrix

$$[A, I_n] = [A, e_1, e_2 \ldots, e_n] \tag{14.4}$$

the simultaneous augmented matrix of the simultaneous system (14.3).

Applying Gauss's method to the kth system of (14.3) and taking into account that now $r(A) = n$ we get by Theorem 14.1 an equivalent system of the form

$$\begin{array}{r}
a_{11}^{(n)} x_1 + a_{12}^{(n)} x_2 + \ldots + a_{1n}^{(n)} x_n = b_{1k}^{(n)} \\
a_{22}^{(n)} x_2 + \ldots + a_{2n}^{n} x_n = b_{2k}^{(n)} \\
\cdot \qquad \cdots \qquad \cdot \qquad\qquad \\
a_{nn}^{(n)} x_n = b_{nk}^{(n)} \\
(k = 1, 2, \ldots, n)
\end{array} \tag{14.5}$$

where $a_{ii}^{(n)} \neq 0$ $(i = 1, 2, \ldots, n)$. It is easy to see that because of the regularity of A no elementary column operations are needed in deriving (14.5) and further that the elementary row operations performed in successive steps of Gauss's method do not depend on k, i.e. the elementary row operations dictated by Gauss's method are the same on all augmented matrices $[A, e_k]$ $(k = 1, 2, \ldots, n)$. This means that Gauss's method can be performed directly by elementary row operations on the $n \times 2n$ simultaneous augmented matrix (14.4). Having arrived at (14.5), subtract the $a_{ji}^{(n)}/a_{ii}^{(n)}$ multiple of the ith equation from the jth equation $(i = n, n - 1, \ldots, 2$; and for fixed $i : j =$

$1, 2, \ldots, i-1$; starting with $i = n$). Then divide the ith equation by $a_{ii}^{(n)}$, $(i = 1, 2, \ldots, n)$. Thus we obtain the following system equivalent to (14.3) (denoting the resulting constants on the right-hand side by $c_{1k}, \ldots, c_{nk}$):

$$\begin{aligned} x_1 &= c_{1k} \\ x_2 &= c_{2k} \\ &\vdots \\ x_n &= c_{nk}\,, \qquad (k = 1, 2, \ldots, n), \end{aligned}$$

or

$$I_n\, x = c_k \tag{14.6}$$

i.e. we have the required solution on the right-hand side. The steps performed in this second stage again do not depend on k, i.e. they are to be performed simultaneously as elementary row operations on the simultaneous augmented matrix (14.4).

Thus, applying the modified Gauss' method as described above, we get from the simultaneous systems (14.3) the equivalent simultaneous systems (14.6). That is to say, by elementary row operations we get from the simultaneous augmented matrix $[A, I_n]$ the simultaneous augmented matrix $[I_n, c_1, c_2, \ldots, c_n]$. However, by (13.10)

$$[I_n, c_1, \ldots, c_n] = [I_n, A^{-1}]\,.$$

15. The Set of Solutions

In studying the structure of the solution set of linear systems it is worth discussing separately those special systems whose constant vector is zero. The linear system

$$Ax = O \tag{15.1}$$

where $A = [a_{ik}]$ is a matrix of type $m \times n$, O is the m-dimensional zero column vector, and the n-dimensional column vector $x = \mathrm{col}\,[x_1, \ldots, x_n]$ is the unknown, is called a *homogeneous linear system*. If in (13.2) $b \neq O$, we call (13.2) an *inhomogeneous linear system*. Everything said in the previous two sections is valid for homogeneous systems as well as for inhomogeneous ones.

The augmented matrix of the homogeneous system (15.1) is $[A, O]$ where O is the m-dimensional zero column vector. Since the zero vector is an element of any subspace it is also an element of the column space of A. Hence $r([A, O]) = r(A)$, i.e. by Theorem 13.2 a homogeneous system always has a solution. In fact, it is quite clear that $x = O$ (here O is the n-dimensional zero column vector) is a solution of (15.1). This is called the *trivial solution* of the homogeneous system (15.1).

Theorem 15.1. The homogeneous linear system (15.1) has non-trivial solutions if and only if $r(A) < n$.

Proof. The statement follows readily from Theorem **13.3** and Corollary 14.2. !

In the special case when the number of unknowns (the dimension of x) is equal to the number of equations we get from the previous theorem the simple but frequently applied Corollary which follows.

Corollary 15.2. Assume that the coefficient matrix A in (15.1) is a square matrix of order n (i.e. $m = n$); the homogeneous system (15.1) has non-trivial solutions if and only if A is singular, i.e. $\det A = 0$.

Returning to the general case, we have

Theorem 15.3. Let $r = r(A)$ and denote the set of solutions of (15.1) by $\mathfrak{M}^{n-r}$; $\mathfrak{M}^{n-r}$ is an $(n - r)$-dimensional subspace of the linear space $\mathcal{Q}^n$ of n-dimensional column vectors.

Proof. It is easy to see that $\mathfrak{M}^{n-r}$ is a subspace. Since if $x_{(1)}$ and $x_{(2)}$ are two arbitrary solutions and λ is a scalar number, then

$$A(x_{(1)} + x_{(2)}) = Ax_{(1)} + Ax_{(2)} = O + O = O,$$

$A(\lambda x_{(1)}) = \lambda(Ax_{(1)}) = \lambda O = O$, and thus $x_{(1)} + x_{(2)}$ and $\lambda x_{(1)}$ are also solutions (i.e. belong to $\mathfrak{M}^{n-r}$).

We have to prove that the subspace $\mathfrak{M}^{n-r}$ is $(n - r)$-dimensional. Applying Gauss's method to the homogeneous system (15.1) we get an equivalent system of the form (14.2), but now on the right-hand side all $b_i^{(r)} = 0$, $(i = 1, 2, \ldots, m)$. Subtract the $a_{ji}^{(r)}/a_{ii}^{(r)}$ multiple of the ith equation from the jth equation ($i = r, r - 1, \ldots, 2$, and for fixed $i : j = 1, 2, \ldots, i - 1$, starting with $i = r$). Then divide the ith equation by $a_{ii}^{(r)}$, $(i = 1, 2, \ldots, r)$. Thus we obtain the following system equivalent to (15.1):

$$\begin{aligned}
x_1 \qquad\qquad & - (c_{11} x_{r+1} + c_{21} x_{r+2} + \ldots + c_{n-r,1} x_n) = 0 \\
x_2 \qquad & - (c_{12} x_{r+1} + c_{22} x_{r+2} + \ldots + c_{n-r,2} x_n) = 0 \\
& \quad \cdot \qquad\quad \cdot \qquad \cdots \quad \cdot \\
x_r & - (c_{1r} x_{r+1} + c_{2r} x_{r+2} + \ldots + c_{n-r,r} x_n) = 0.
\end{aligned} \tag{15.2}$$

Here the coefficient of x_{r+i} in the jth equation has been denoted by $-c_{ij}$ for convenience's sake ($j = 1, 2, \ldots, r$; $i = 1, 2, \ldots, n - r$). In the solution of this system the values of $x_{r+1}, x_{r+2}, \ldots, x_n$ can be chosen arbitrarily. Let us substitute first

$$x_{r+1} = 1, \qquad x_{r+2} = \ldots = x_n = 0,$$

then

$$x_{r+1} = 0,\ x_{r+2} = 1,\ x_{r+3} = \ldots = x_n = 0,$$

and so on; finally

$$x_{r+1} = \ldots = x_{n-1} = 0, \; x_n = 1.$$

The corresponding solutions of (15.1) are, in view of (15.2),

$$c_{r+1} = \begin{bmatrix} c_{11} \\ c_{12} \\ \vdots \\ c_{1r} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad c_{r+2} = \begin{bmatrix} c_{21} \\ c_{22} \\ \vdots \\ c_{2r} \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, c_n = \begin{bmatrix} c_{n-r,1} \\ c_{n-r,2} \\ \vdots \\ c_{n-r,r} \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

respectively. We know from Example 9.2 that these column vectors are linearly independent. This means that $\mathfrak{M}^{n-r}$ contains $(n - r)$ linearly independent column vectors. We shall now show that these column vectors span $\mathfrak{M}^{n-r}$, that is, any $x \in \mathfrak{M}^{n-r}$ can be expressed as a linear combination of $c_{r+1}, \ldots, c_n$. Let $x = \text{col}\,[x_1, \ldots, x_n]$ be an arbitrary solution and substitute x into (15.2). We then have

$$\begin{aligned} x_1 &= x_{r+1}\,c_{11} + x_{r+2}\,c_{21} + \ldots + x_n\,c_{n-r,1} \\ x_2 &= x_{r+1}\,c_{12} + x_{r+2}\,c_{22} + \ldots + x_n\,c_{n-r,2} \\ &\cdot \quad \cdot \quad \cdot \quad \cdots \quad \cdot \\ x_r &= x_{r+1}\,c_{1r} + x_{r+2}\,c_{2r} + \ldots + x_n\,c_{n-r,r}\,. \end{aligned}$$

We attach the following equalities to the preceding ones:

$$\begin{aligned} x_{r+1} &= x_{r+1} \\ x_{r+2} &= \qquad\qquad x_{r+2} \\ \cdot & \qquad\qquad\quad \cdot \\ x_n &= \qquad\qquad\qquad\qquad x_n\,. \end{aligned}$$

If we look at the combined set of these equalities we see that

$$x = x_{r+1}\,c_{r+1} + x_{r+2}\,c_{r+2} + \ldots + x_n\,c_n\,.$$

Applying Theorem 9.12 it follows that $\mathfrak{M}^{n-r}$ is a subspace of dimension $(n - r)$. !

Corollary 15.4. Let $r = r(A)$ and assume that $x_{(1)}, x_{(2)}, \ldots, x_{(n-r)}$ are linearly independent solutions of (15.1); then any solution x of (15.1) can be expressed uniquely as a linear combination of $x_{(1)}, \ldots, x_{(n-r)}$:

$$x = \sum_{k=1}^{n-r} \lambda_k\,x_{(k)}\,. \tag{15.3}$$

Proof. The statement follows directly from Theorem 9.11, since $x_{(1)}, \ldots, x_{(n-r)}$ form a base of the subspace $\mathfrak{M}^{n-r}$ of the solutions. !

If $r = r(A)$ in (15.1) and $x_{(1)}, \ldots, x_{(n-r)}$ are linearly independent solutions, then the expression

$$\sum_{k=1}^{n-r} \lambda_k x_{(k)},$$

where λ_k $(k = 1, 2, \ldots, n-r)$ may assume arbitrary values, is called the *general solution* of (15.1), since the set

$$\mathfrak{M}^{n-r} = \left\{x \in \mathfrak{Q}^n : x = \sum_{k=1}^{n-r} \lambda_k x_{(k)}, (\lambda_1, \ldots, \lambda_{n-r}) \in \mathfrak{R}^{n-r} \text{ or } \mathfrak{C}^{n-r}\right\}$$

contains all solutions of (15.1). The base $x_{(1)}, \ldots, x_{(n-r)}$ of the subspace $\mathfrak{M}^{n-r}$ of the solutions is called a *fundamental system* of (15.1).

Considering now the inhomogeneous linear system (13.2), the system (15.1) (with the same coefficient matrix) will be called the homogeneous system corresponding to (13.2).

THEOREM 15.5. If the column vectors $x_{(1)}$ and $x_{(2)}$ are solutions of (13.2) then their difference $(x_{(1)} - x_{(2)})$ is a solution of the corresponding homogeneous system (15.1).

Proof.

$$A(x_{(1)} - x_{(2)}) = Ax_{(1)} - Ax_{(2)} = b - b = O\,.\,!$$

THEOREM 15.6. If x_0 is a given solution of (13.2), then for each solution x of (13.2) one can find a solution x_H of the corresponding homogeneous system (15.1) for which

$$x = x_0 + x_H\,.$$

Proof. Applying the previous theorem to $x - x_0$ we see that this difference is an x_H solution of (15.1). !

If x_0 is a given solution of (13.2) and $\sum_{k=1}^{n-r} \lambda_k x_{(k)}$ is the general solution of (15.1), then

$$x_0 + \sum_{k=1}^{n-r} \lambda_k x_{(k)},$$

where λ_k $(k = 1, 2, \ldots, n-r)$ may assume arbitrary values, is called the *general solution* of (13.2), since the set

$$\left\{x \in \mathfrak{Q}^n : x = x_0 + \sum_{k=1}^{n-r} \lambda_k x_{(k)}, (\lambda_1, \ldots, \lambda_{n-r}) \in \mathfrak{R}^{n-r} \text{ or } \mathfrak{C}^{n-r}\right\}$$

contains all solutions of (13.2).

As an application of the previous result the important ideas of eigenvectors and eigenvalues of quadratic matrices are introduced. We use these concepts in Chapter Six.

DEFINITION 15.1. Consider the quadratic matrix A of order n; we say that the scalar λ and the n-dimensional column vector s are an *eigenvalue* and a corresponding *eigenvector*, respectively, of A if $s \neq O$ and

$$As = \lambda s \, . \tag{15.4}$$

THEOREM 15.7. The scalar number λ is an eigenvalue of the quadratic matrix A (an eigenvector corresponding to λ exists) if and only if λ is a root of the polynomial equation

$$\det\,(A - \lambda I) = 0. \tag{15.5}$$

Proof. By the defining condition (15.4) λ is an eigenvalue if and only if an $s \neq O$ column vector exists such that $As - \lambda s = O$, that is

$$(A - \lambda I)\, s = O \, . \tag{15.6}$$

(15.6) is a homogeneous linear system for the unknown coordinates of s and it has, by Corollary 15.2, a non-trivial solution if and only if its coefficient matrix $A - \lambda I$ is singular, i.e. if and only if λ satisfies (15.5). !

If A is a matrix of order n (then I is, of course, the unit matrix of order n as well), then det $(A - \lambda I)$ on the left-hand side of (15.5) is a polynomial of degree n and is called the *characteristic polynomial* of A. Equation (15.5) is called the *characteristic equation* of A.

Remember that a polynomial with real coefficients may have complex roots, and so a matrix with real elements may have complex eigenvalues and, consequently, eigenvectors with complex elements. It follows from the Fundamental Theorem of Algebra (Theorem 7.1) that any matrix has at least one complex eigenvalue and one complex eigenvector. In Section 23 we shall return to the important problem of finding conditions under which all eigenvalues and eigenvectors are real.

16. A Basic Problem of Linear Programming

As an application of the foregoing theory we deal in this Section with one of the most common problems of modern industry and economics: linear programming. The scope of this introductory book does not allow a thorough general treatment of the problem and the methods applied. Only a simplified special problem will be dealt with, but the treatment allows of generalizations and is characteristic of methods applied in other cases.

Example 16.1. Assume that a company is using m kinds of resources (raw materials, manpower, electric power, etc.) and is producing n kinds of products. Denote by a_{ik} the amount of the ith resource used in producing one unit of the kth product ($i = 1, 2, \ldots, m$; $k = 1, 2, \ldots, n$, cf. Example 8.1), by b_i the total available amount of the ith resource ($i = 1, 2, \ldots, m$), by c_k the price of a unit of the kth product ($k = 1, 2, \ldots, n$), and by x_k the (at present unknown) amount to be produced of the kth product ($k = 1, 2, \ldots, n$).

The problem is to find the values of $x_1, x_2, \ldots, x_n$ that maximize the income of the company. Introducing the row vector $c' = [c_1, \ldots \ldots, c_n]$ and the column vector $x = \operatorname{col}[x_1, \ldots, x_n]$, we get the income from

$$c' x = \sum_{k=1}^{n} c_k x_k . \tag{16.1}$$

This expression is called the *objective function*. If the amount x_k is produced from the kth product ($k = 1, 2, \ldots, n$ and obviously $x_k \geq 0$), then the amount of the ith resource used is

$$\sum_{k=1}^{n} a_{ik} x_k$$

and this amount cannot be higher than b_i, ($i = 1, 2, \ldots, m$). This means that the objective function (16.1) is to be maximized under the *constraints*

$$\sum_{k=1}^{n} a_{ik} x_k \leq b_i \qquad (i = 1, 2, \ldots, m),$$

$$x_k \geq 0 \qquad (k = 1, 2, \ldots, n) .$$

The practical problem described in the previous example can easily be reduced, by introducing some auxiliary variables, to the following problem. Let there be a real $m \times n$ matrix $A = [a_{ik}]$, a row vector $c' = [c_1, \ldots, c_n]$, a column vector $b = \operatorname{col}[b_1, \ldots, b_m]$, and denote by $x = \operatorname{col}[x_1, \ldots, x_n]$ the column vector of the variables. Determine the maximum of the objective function (16.1) under the constraints (i.e. conditions)

$$Ax = b, \; x \geq 0 \tag{16.2}$$

where $x \geq 0$ is a shorthand notation for $x_k \geq 0$ ($k = 1, 2, \ldots n$). We now introduce some other notations. The column vectors of the matrix A are denoted by $a_1, \ldots, a_n$ and we assume that the rank of A is r, i.e. $r = r(A)$. The solutions of the linear system $Ax = b$ satisfying the non-negativity conditions $x \geq 0$, i.e. the solutions of (16.2), are called *feasible solutions* and their set is denoted by $\mathcal{M}$:

$$\mathcal{M} = \{x \in \mathcal{Q}^n : Ax = b, \; x \geq 0\}.$$

A base of the column space $\mathcal{G}^r$ of the matrix A (see Theorem 12.5), i.e. a set of r linearly independent columns $a_{i_1}, a_{i_2}, \dots, a_{i_r}$ of A, is called a *basis* if there exists a feasible solution $x = \mathrm{col}\,[x_1, \dots, x_n] \in \mathcal{M}$ with $x_j = 0$, $j \neq i_k$ $(k = 1, 2, \dots, r)$. We say then that this x is a *basic solution* corresponding to the given basis. The basis is called *degenerate* if at least one of the elements x_{i_k} of the basic solution corresponding to it is also zero. Thus a basic solution corresponding to the given basis looks like

$$x = \mathrm{col}\,[0, 0, \dots, x_{i_1}, 0, \dots, 0, x_{i_2}, \dots, x_{i_r}, 0, \dots, 0]$$

nd the basis is degenerate if in some of the i_1th, i_2th, ..., i_rth places we have a zero. The basis and its basic solution are called *optimal* if the objective function (16.1) assumes its maximum at the corresponding basic solution, this time denoted by v, that is if

$$c'\,v = \max_{x \in \mathcal{M}} c'\,x\,.$$

It is easy to see that a given basis determines the corresponding basic solution uniquely, or in other words, one and only one basic solution corresponds to each basis. If $a_{i_1}, \dots, a_{i_r}$ is a basis then these vectors form a base of the column space $\mathcal{G}^r$ and $b \in \mathcal{G}^r$. However, by Theorem 9.11, b is a unique linear combination of $a_{i_1}, \dots, a_{i_r}$.

In what follows we assume that (16.2) has a solution, that is to say the set $\mathcal{M}$ is not empty. (If this were not so then, obviously, no maximum would exist.) We give an algorithm that leads to an optimal basic solution provided that a solution of (16.2) is known and some sufficient conditions are fulfilled. In the first stage (first and second theorems) (16.2) will be considered. We turn to the objective function only in the second stage.

Theorem 16.1. If (16.2) has a solution then it also has a basic solution.

Proof. If $y = \mathrm{col}\,[y_1, \dots, y_n]$ is a solution of (16.2), we may assume without loss of generality that the first l elements of y are positive and the last $(n - l)$ elements are zero:

$$a_1 y_1 + a_2 y_2 + \dots + a_l y_l = b\,. \tag{16.3}$$

If the column vectors $a_1, \dots, a_l$ are linearly independent and $l = r$, then y is a basic solution (if $l < r$, then y is a degenerate basic solution). Assume that $a_1, a_2, \dots, a_l$ are linearly dependent. Then there are real numbers $\lambda_1, \lambda_2, \dots, \lambda_l$ such that $|\lambda_1| + \dots + |\lambda_l| > 0$,

$$a_1 \lambda_1 + a_2 \lambda_2 + \dots + a_l \lambda_l = 0, \tag{16.4}$$

and some of the λ_i are positive (if this were not so we could multiply (16.4) by (-1)). Let

$$\frac{y_j}{\lambda_j} = \min_{\lambda_i > 0} \frac{y_i}{\lambda_i} \tag{16.5}$$

and express the column vector a_j from (16.4):

$$a_j = - \sum_{\substack{i=1 \\ i \neq j}}^{l} \frac{\lambda_i}{\lambda_j} a_i.$$

Substituting this expression into (16.3) for a_j we get

$$\begin{aligned} a_1 \left(y_1 - y_j \frac{\lambda_1}{\lambda_j}\right) + \ldots + a_{j-1} \left(y_{j-1} - y_j \frac{\lambda_{j-1}}{\lambda_j}\right) \\ + a_{j+1} \left(y_{j+1} - y_j \frac{\lambda_{j+1}}{\lambda_j}\right) + \ldots + a_l \left(y_l - y_j \frac{\lambda_l}{\lambda_j}\right) = b\,. \qquad (16.6) \end{aligned}$$

Because of the choice (16.5) all the coefficients on the left-hand side are non-negative and thus the column vector

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_{j-1} \\ u_j \\ u_{j+1} \\ \vdots \\ u_l \\ u_{l+1} \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} y_1 - y_j \dfrac{\lambda_1}{\lambda_j} \\ \vdots \\ y_{j-1} - y_j \dfrac{\lambda_{j-1}}{\lambda_j} \\ 0 \\ y_{j+1} - y_j \dfrac{\lambda_{j+1}}{\lambda_j} \\ \vdots \\ y_l - y_j \dfrac{\lambda_l}{\lambda_j} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

is a solution of (16.2) and has $(l - 1)$ positive (non-zero) elements at most.

If the column vectors $a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_l$ are linearly dependent then we repeat the procedure. There are real numbers $\mu_1, \mu_2, \ldots$ $\ldots, \mu_{j-1}, \mu_{j+1}, \ldots, \mu_l$ such that some of them are positive and

$$a_1 \mu_1 + a_2 \mu_2 + \ldots + a_{j-1} \mu_{j-1} + a_{j+1} \mu_{j+1} + \ldots a_l \mu_l = 0. \qquad (16.7)$$

Let

$$\frac{u_h}{\mu_h} = \min_{\mu_i > 0} \frac{u_i}{\mu_i},$$

express the vector a_h from (16.7) and substitute its expression into

(16.6). Thus we obtain a solution of (16.2) having $(l-2)$ positive (non-zero) elements at most. The procedure can obviously be continued and the column vectors can be eliminated one by one from the set a_i, $(i = 1, 2, \ldots, l)$ until the remaining ones are already linearly independent. Then according to what was said at the beginning of the proof we have arrived at a basic solution. !

Starting from a solution the procedure given in the previous proof leads to a basic solution. Therefore, from now on we restrict our attention to basic solutions. First we show how starting from one basis another basis can be introduced by simply exchanging one of the column vectors belonging to the first basis for another column vector of A.

THEOREM 16.2. Let, for instance, $a_1, \ldots, a_r$ be a basis, $x = \operatorname{col}[x_1, \ldots, x_r, 0, \ldots, 0]$ the corresponding basic solution:

$$a_1 x_1 + a_2 x_2 + \ldots + a_r x_r = b, \qquad x \geq O \tag{16.8}$$

and express all column vectors of A in terms of the given basis:

$$a_h = a_1 \lambda_{1h} + a_2 \lambda_{2h} + \ldots + a_r \lambda_{rh} \quad (h = 1, 2, \ldots, n); \tag{16.9}$$

if for some fixed k $(k > r)$ there are positive numbers among the λ_{ik} $(i = 1, 2, \ldots, r)$ and

$$\frac{x_j}{\lambda_{jk}} = \min_{\lambda_{ik}>0} \frac{x_i}{\lambda_{ik}} \tag{16.10}$$

then

$$a_1, a_2, \ldots, a_{j-1}, a_k, a_{j+1}, \ldots, a_r \tag{16.11}$$

is a basis.

Proof. The column vectors (16.11) are linearly independent. Assume, contrary to the statement, that there are $\mu_1, \ldots, \mu_{j-1}, \mu_k, \mu_{j+1}, \ldots, \mu_r$ not all of them zero for which

$$\mu_1 a_1 + \ldots + \mu_{j-1} a_{j-1} + \mu_k a_k + \mu_{j+1} a_{j+1} + \ldots + \mu_r a_r = 0. \tag{16.12}$$

If μ_k were zero then the column vectors $a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_r$ would be linearly dependent, contrary to the assumption (cf. Corollary 9.3). Thus $\mu_k \neq 0$ and a_k can be expressed from (16.12) as a linear combination of the rest of the column vectors. However, the decomposition of a_k with respect to the basis a_i $(i = 1, 2, \ldots, r)$ is uniquely determined, and hence the coefficients in the expression for a_k from (16.12) must be equal to the respective coefficients in (16.9) $(h = k)$. In particular $\lambda_{jk} = 0$ must hold, contrary to (16.10). As this is a contradiction, the column vectors (16.11) are linearly independent.

Now we show that the column vectors (16.11) form not only a base of the column space $\mathcal{G}^r$ but a basis. Expressing a_j from (16.9) $(h = k)$:

$$a_j = \frac{1}{\lambda_{jk}} a_k - \sum_{\substack{i=1 \\ i \neq j}}^{r} \frac{\lambda_{ik}}{\lambda_{jk}} a_i \,. \tag{16.13}$$

Substituting this expression into (16.8) we get

$$\sum_{\substack{i=1\\ i\neq j}}^{r} a_i \left(x_i - x_j \frac{\lambda_{ik}}{\lambda_{jk}}\right) + \frac{x_j}{\lambda_{jk}} a_k = b\,. \tag{16.14}$$

Thus the column vector

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_{j-1} \\ u_j \\ u_{j+1} \\ \vdots \\ u_r \\ u_{r+1} \\ \vdots \\ u_k \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} x_1 - x_j \dfrac{\lambda_{1k}}{\lambda_{jk}} \\ \vdots \\ x_{j-1} - x_j \dfrac{\lambda_{j-1,k}}{\lambda_{jk}} \\ 0 \\ x_{j+1} - x_j \dfrac{\lambda_{j+1,k}}{\lambda_{jk}} \\ \vdots \\ x_r - x_j \dfrac{\lambda_{rk}}{\lambda_{jk}} \\ 0 \\ \vdots \\ \dfrac{x_j}{\lambda_{jk}} \\ \vdots \\ 0 \end{bmatrix} \tag{16.15}$$

is a solution of the system $Ax = b$. At the same time u is a feasible solution if $\lambda_{ik} > 0$, from (16.10). If $\lambda_{ik} \leq 0$ then obviously

$$u_i = x_i - x_j \frac{\lambda_{ik}}{\lambda_{jk}} \geq 0 \qquad (i = 1, 2, \ldots, r;\ i \neq j)$$

$(x_j \geq 0,\ \lambda_{jk} > 0)$, $u_k \geq 0$ and $u_i = 0$, $(i = j,\ i = r + 1,\ r + 2, \ldots n,\ i \neq k)$. Hence (16.11) is a basis. !

Note that substituting (16.13) into (16.9) we get the column vectors of A in terms of the new basis:

$$a_h = \sum_{\substack{i=1\\ i\neq j}}^{r} a_i \lambda_{ih} + \frac{\lambda_{jh}}{\lambda_{jk}} a_k - \lambda_{jh} \sum_{\substack{i=1\\ i\neq j}}^{r} \frac{\lambda_{ik}}{\lambda_{jk}} a_i$$

$$= \sum_{\substack{i=1\\ i\neq j}}^{r} a_i \left(\lambda_{ih} - \lambda_{jh}\frac{\lambda_{ik}}{\lambda_{jk}}\right) + \frac{\lambda_{jh}}{\lambda_{jk}} a_k$$

$$(h = 1, 2, \ldots, n). \qquad (16.16)$$

Expression (16.14) for b can be made analogous to (16.16) by introducing the notation

$$a_0 = b, \ \lambda_{i0} = x_i \quad (i = 1, 2, \ldots, r);$$

$$b = a_0 = \sum_{i=1}^{r} a_i \left(\lambda_{i0} - \lambda_{j0}\frac{\lambda_{ik}}{\lambda_{jk}}\right) + \frac{\lambda_{j0}}{\lambda_{jk}} a_k . \qquad (16.17)$$

Now we give a criterion that ensures the increase of the objective function by a change of basis (performed according to the previous theorem).

THEOREM 16.3. Under the assumptions and with the notation of Theorem 16.2 let us change from the basis $a_1, \ldots, a_r$ to the basis (16.11), the basic solution corresponding to the latter being u given by (16.15); if

$$d_k = \sum_{i=1}^{r} \lambda_{ik} c_i - c_k < 0 , \qquad (16.18)$$

then

$$c' u \geq c' x .$$

Proof.

$$c' u = \sum_{i=1}^{r} c_i \left(x_i - x_j \frac{\lambda_{ik}}{\lambda_{jk}}\right) + c_k \frac{x_j}{\lambda_{jk}}$$

$$= c' x - \frac{x_j}{\lambda_{jk}} \left[\sum_{i=1}^{r} \lambda_{ik} c_i - c_k\right] \geq c' x$$

because of (16.18) since $x_j \geq 0$, $\lambda_{jk} \geq 0$. !

We have achieved, in principle, the aim of increasing the value of the objective function by changing a basis for another. A criterion is needed that tells us where to stop.

THEOREM 16.4. Let, for instance, $a_1, \ldots, a_r$ be a basis and

$$a_k = \sum_{i=1}^{r} \lambda_{ik} a_i \quad (k = 1, 2, \ldots, n); \qquad (16.19)$$

if for all $k = 1, 2, \ldots, n$

$$d_k = \sum_{i=1}^{r} \lambda_{ik} c_i - c_k \geq 0 \qquad (16.20)$$

then this basis is optimal.

Note that if $k \leq r$ then $\lambda_{ik} = \delta_{ik}$ (the Kronecker symbol) and as a consequence $d_k = 0$.

Proof. We show that the value of the objective function at an arbitrary feasible solution $y = \operatorname{col}[y_1, \ldots, y_n]$ is not greater than at the basic solution $v = \operatorname{col}[v_1, \ldots, v_n]$ corresponding to the basis $a_1, \ldots, a_r$. Applying (16.19) we have

$$\begin{aligned} b = \sum_{k=1}^{n} y_k a_k &= \sum_{k=1}^{r} y_k a_k + \sum_{k=r+1}^{n} \left(y_k \sum_{i=1}^{r} \lambda_{ik} a_i \right) \\ &= \sum_{i=1}^{r} y_i a_i + \sum_{i=1}^{r} \left(\sum_{k=r+1}^{n} \lambda_{ik} y_k \right) a_i = \sum_{i=1}^{r} \left(y_i + \sum_{k=r+1}^{n} y_k \lambda_{ik} \right) a_i . \end{aligned}$$

On the other hand

$$b = \sum_{i=1}^{r} v_i a_i .$$

Since the decomposition of b with respect to the given basis is unique we get, comparing the last two expressions,

$$v_i = y_i + \sum_{k=r+1}^{n} y_k \lambda_{ik} , \qquad (i = 1, 2, \ldots, r).$$

The value of the objective function at v is

$$\begin{aligned} c' v = \sum_{i=1}^{r} v_i c_i &= \sum_{i=1}^{r} \left(y_i + \sum_{k=r+1}^{n} y_k \lambda_{ik} \right) c_i \\ &= \sum_{i=1}^{n} y_i c_i - \sum_{k=r+1}^{n} y_k c_k + \sum_{k=r+1}^{n} \left(y_k \sum_{i=1}^{r} \lambda_{ik} c_i \right) \\ &= c' y + \sum_{k=r+1}^{n} y_k d_k \geq c' y , \end{aligned}$$

since $y_k \geq 0$, $d_k \geq 0$ $(k = r + 1, r + 2, \ldots, n)$. !

Finally we give a sufficient criterion that tells us if the maximization problem has no solution. This happens, provided that feasible solutions exist, if the objective function is not bounded from above on the set $\mathfrak{M}$ of feasible solutions.

THEOREM 16.5. Let, for instance, $a_1, \ldots, a_r$ be a basis, assume that for some fixed k $(k > r)$

$$a_k = \sum_{i=1}^{r} \lambda_{ik} a_i , \quad d_k = \sum_{i=1}^{r} \lambda_{ik} c_i - c_k ,$$

d_k is negative and, for all $i = 1, 2, \ldots, r$, we have $\lambda_{ik} \leq 0$; then the objective function is not bounded from above on the set $\mathfrak{M}$ of feasible solutions.

Proof. Let p be an arbitrary positive number. It will be shown that the objective function assumes values greater than p on the set $\mathcal{M}$. Consider an arbitrary positive number q, and let $x = \operatorname{col}[x_1, \ldots, x_n]$ denote the basic solution corresponding to the given basis. Since

$$qa_k = q \sum_{i=1}^{r} \lambda_{ik} a_i ,$$

i.e.

$$qa_k - \sum_{i=1}^{r} q\lambda_{ik} a_i = 0 ,$$

we have

$$\begin{aligned} b = \sum_{i=1}^{r} a_i x_i &= \sum_{i=1}^{r} a_i x_i + qa_k - \sum_{i=1}^{r} q\lambda_{ik} a_i \\ &= \sum_{i=1}^{r} (x_i - q\lambda_{ik}) a_i + qa_k . \end{aligned}$$

Taking into account that $\lambda_{ik} \leq 0$ $(i = 1, 2, \ldots, r)$ and $q > 0$ we see that the n-dimensional column vector

$$u = \left[\begin{array}{c} x_1 - q\,\lambda_{1k} \\ x_2 - q\,\lambda_{2k} \\ \vdots \\ x_r - q\,\lambda_{rk} \\ 0 \\ \vdots \\ q \\ \vdots \\ 0 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ \\ \\ (k \\ \\ \\ \end{array}$$

is a feasible solution. But

$$c'u = c'x - q\left[\sum_{i=1}^{r} \lambda_{ik} c_i - c_k\right] = c'x - qd_k .$$

However, $c'x - qd_k \geq p$ provided that q is chosen in the following way $(-d_k > 0)$:

$$q > \frac{p - c'x}{-d_k} .$$

Hence $c'u \geq p$.!

The results of the previous theorems show that we have a method leading to the solution of the maximization problem. Firstly we must solve (16.2) and find a solution of the linear system satisfying also the non-negativity conditions. Being in possession of such a feasible solution we have an algorithm that leads to a basis and a basic solution. Then changing from basis to basis by exchanging a column vector of the basis for a suitable column of A we get an increasing sequence of objective function values. At each step we may check whether or not we have reached the optimal solution.

Because of geometrical considerations this method is called the *simplex method*. The method can be improved and it can be shown that one can avoid with care the occurrence of the same basis twice, the so-called 'cycling'. If the objective function is bounded on the set of feasible solutions it can be proved that it assumes its maximum at a basic solution. Since there are a finite number of bases, if the problem has a solution we reach it in a finite number of steps. (See G. Hadley: Linear Programming, Addison–Wesley, 1962.)

The following example illustrates the method. The so-called 'simplex table' is introduced in this example as an aid to calculation.

EXAMPLE 16.2. Determine the maximum of the objective function $c'x = x_2 + x_4$ under the constraints

$$\begin{aligned} 3x_1 + x_2 + 7x_3 + 2x_4 &= 16 \\ 2x_1 + x_2 + 5x_3 + x_4 &= 11 \\ x_i &\geq 0 \qquad (i = 1, 2, 3, 4). \end{aligned}$$

Thus $c' = [0, 1, 0, 1]$, x is a four-dimensional column vector, and the matrix A and the column vector b of the theory are

$$A = \begin{bmatrix} 3 & 1 & 7 & 2 \\ 2 & 1 & 5 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 16 \\ 11 \end{bmatrix}.$$

It is easy to get a feasible solution of the linear system. For example, with $y = \text{col}\,[2, 1, 1, 1]$ we have $Ay = b$ and obviously $y \geq 0$. Since a feasible solution exists there is also a basic solution (see Theorem 16.1). Consider the equation

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} \lambda_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \lambda_2 + \begin{bmatrix} 7 \\ 5 \end{bmatrix} \lambda_3 + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \lambda_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

corresponding to (16.4). A solution of the latter is, for example,

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (2, 1, -1, 0)$$

$$\min_{\lambda_i > 0} \frac{y_i}{\lambda_i} = \min \left\{ \frac{2}{2}, \frac{1}{1} \right\} = \frac{y_1}{\lambda_1} = \frac{y_2}{\lambda_2} = 1 .$$

Let us choose $j = 2$ (in (16.5)). Thus we eliminate the second column vector a_2 of the matrix A. The new feasible solution is, according to (16.6),

$$x = \operatorname{col}\left[2 - 1\,\frac{2}{1},\ \ 0,\ \ 1\ \ -1\ \ \frac{-1}{1},\ \ 1\ \ -1\ \ \frac{0}{1}\right]$$

$$= \operatorname{col}\,[0,\ 0,\ 2,\ 1].$$

Since $a_3 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$ and $a_4 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ are linearly independent, x is a basic solution.

If $c_0 = 0$, then the formula (16.18) for $k = 0$ yields the value of the objective function at the basic solution in question:

$$d_0 = c'x$$

in this case. Taking this into account and applying the notation introduced in (16.9), (16.17) and (16.18), we get the following simplex table:

			c_1	c_2	c_3	c_4
		a_0	a_1	a_2	a_3	a_4
c_3	a_3	λ_{30}	λ_{31}	λ_{32}	λ_{33}	λ_{34}
c_4	a_4	λ_{40}	λ_{41}	λ_{42}	λ_{43}	λ_{44}
		d_0	d_1	d_2	d_3	d_4

The table corresponding to the basis a_3, a_4 is in this case

			0	1	0	1
		b	a_1	a_2	a_3	a_4
0	a_3	2	1/3	1/3	1	0
1	a_4	1	1/3	− 2/3	0	1
		1	1/3	− 5/3	0	0

Here $d_2 < 0$, $\lambda_{32} > 0$ and

$$\frac{x_3}{\lambda_{32}} = \min_{\lambda_{i2}>0}\left\{\frac{x_i}{\lambda_{i2}}\right\} = \min\left\{\frac{x_3}{\lambda_{32}}\right\}.$$

Thus applying Theorems 16.2 and 16.3 we may set $j = 3$, $k = 2$, i.e. exchange the column vector a_3 for a_2 while the value of the objective function will increase (will not decrease). Applying formulae (16.16) and (16.17) we get from the previous table the simplex table corresponding to the basis a_2, a_4:

			0	1	0	1
		b	a_1	a_2	a_3	a_4
1	a_2	6	1	1	3	0
1	a_4	5	1	0	2	1
		11	2	0	5	0

Here $d_k \geq 0$ for $k = 1, 2, 3, 4$; thus by Theorem 16.4 the basis a_2, a_4 is optimal. It is easy to find the corresponding optimal basic solution $v = \operatorname{col}[0, v_2, 0, v_4]$ from the system $a_2 v_2 + a_4 v_4 = b$:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} v_2 + \begin{bmatrix} 2 \\ 1 \end{bmatrix} v_4 = \begin{bmatrix} 16 \\ 11 \end{bmatrix}.$$

It is $v' = \operatorname{col}[0, 6, 0, 5]$. The maximum of the objective function on the set $\mathfrak{M}$ of feasible solutions is equal to d_0 in the last table;

$$\max_{\mathfrak{M}} (x_2 + x_4) = 11 .$$

Exercises

IV.1 Determine the solution of the system of linear equations:

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_2 + x_3 &= 3 \\ x_3 &= 1. \end{aligned}$$

IV.2 In the following system of linear equations there are less unknowns than equations. Determine the solutions.

$$\text{a)}\quad \begin{aligned} x_1 + x_2 - x_3 &= 3 \\ 2x_1 + x_2 - x_3 &= 5 \\ 2x_1 - x_2 + x_3 &= 3 \\ 5x_1 + x_2 - x_3 &= 11 \end{aligned} \qquad \text{b)}\quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 3x_1 + 2x_3 &= 0 \\ 2x_1 + x_2 + x_3 &= 1/2 \\ 2x_1 + 2x_2 + 3x_3 &= 1 \end{aligned}$$

c)
$$\begin{aligned} x_1 - 8x_2 &= 3 \\ 2x_1 + x_2 &= 1 \\ 4x_1 + 7x_2 &= -4. \end{aligned}$$

IV.3 In the following systems of equations there are more unknowns than equations. Determine the solutions.

a)
$$\begin{aligned} 5x_1 - x_2 + 2x_3 + x_4 &= 7 \\ 2x_1 + x_2 + 4x_3 - 2x_4 &= 1 \\ x_1 - 3x_2 - 6x_3 + 5x_4 &= 0 \end{aligned}$$

b)
$$\begin{aligned} x_1 + x_2 - 2x_3 - x_4 + x_5 &= 1 \\ -x_2 + 7x_3 + 7x_4 + x_5 &= 1. \end{aligned}$$

c) Can you construct a system of this kind having only one solution?

IV.4 Find all the integer solutions of the system of equations

$$\begin{aligned} x_1 - x_2 + x_3 - x_4 &= 0 \\ 2x_2 - 2x_3 + x_4 &= 1 \\ 10x_3 - 5x_4 &= -5. \end{aligned}$$

IV.5 Find all the integer solutions of the system of equations

$$\begin{aligned} x_1 + 7x_2 + 4x_3 &= 0 \\ 3x_1 + x_2 + 12x_3 &= 0 \\ 3x_1 + 7x_2 + 12x_3 &= 0 \\ 3x_1 + 12x_3 &= 0. \end{aligned}$$

IV.6 Solve the following homogeneous system of equations:

$$\begin{aligned} 5x_1 - 2x_2 + x_3 &= 0 \\ x_1 + x_2 + 6x_3 &= 0 \\ 3x_1 + 4x_3 &= 0 \\ -x_2 + 2x_3 &= 0. \end{aligned}$$

IV.7
$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 3 \\ -2 & 1 & 2 \end{bmatrix}$$

$$b_1 = \begin{bmatrix} 8 \\ 9 \\ 6 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad b_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

Solve simultaneously the following systems of equations:

$$Ax = b_1, \; Ax = b_2, \; Ax = b_3 .$$

IV.8 $$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ -1 & 1 & -2 & 1 \\ 2 & -1 & 3 & 1 \end{bmatrix}.$$

Find the inverse of A, by solving systems of linear equations simultaneously.

IV.9 At which value of the parameter a does the following system have solutions?

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + x_4 &= 1 \\ x_1 + 3x_2 + 3x_3 + 4x_4 &= -6 \\ x_1 - 4x_2 - 39x_3 - 17x_4 &= a. \end{aligned}$$

IV.10 Find the values of the parameters a and b for which the following system has more than one solution:

$$\begin{aligned} 3x_1 + 5x_2 - x_3 &= 1 \\ x_1 + ax_2 + 2x_3 &= 2 \\ x_1 + 9x_2 - 5x_3 &= b\,. \end{aligned}$$

IV.11 Given the system of linear equations

$$\begin{aligned} cx_1 + x_2 + x_3 &= 1 \\ x_1 + cx_2 + x_3 &= c \\ x_1 + x_2 + cx_3 &= c^2\,, \end{aligned}$$

find the values of the parameter c for which the system has

a) a unique solution

b) an infinite number of solutions

c) no solution at all.

IV.12 $$\begin{aligned} c^2x_1 + 3x_2 + 2x_3 &= 0 \\ cx_1 - x_2 + 4x_3 &= 0 \\ x_2 + 4x_3 &= 0. \end{aligned}$$

Find the value of the parameter c for which the system has an infinite number of solutions.

IV.13 For which values of u and v is there a solution to the system of equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 14 \\ x_1 + 4x_2 + 9x_3 &= 44 \\ x_1 + 8x_2 + 27x_3 &= 134 \\ -x_1 - 6x_2 - 15x_3 &= u \\ x_1 - 2x_2 - 15x_3 &= v\ ? \end{aligned}$$

IV.14 Can you find values of the parameters u and v for which the following system has more than one solution?

$$x_1 + x_2 + 2x_3 = 0$$

$$\begin{aligned} x_1 - 2x_2 - x_3 &= 0 \\ x_1 + x_2 + x_3 &= u \\ 3x_1 - x_2 + 2x_3 &= u + v. \end{aligned}$$

IV.15 Let A be a second-order non-zero square matrix. Can you find a non-unit matrix B such that

$$AB = A\ ?$$

Check your result for the case

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}.$$

IV.16 Determine the maximum of the objective function

$$c'x = x_1 + 2x_2 + 3x_3 - x_4$$

under the constraints

$$\begin{aligned} x_1 - 3x_2 - x_3 - 2x_4 &= -4 \\ x_1 - x_2 + x_3 &= 0 \\ x_i \geq 0 \quad (i &= 1, 2, 3, 4). \end{aligned}$$

IV.17 Determine the maximum of the objective function

$$c'x = x_1 - 3x_2 - 5x_3 - x_4$$

under the constraints

$$\begin{aligned} x_1 - 3x_2 - x_3 - 2x_4 &= -4 \\ x_1 - x_2 + x_3 &= 0 \\ x_i \geq 0 \quad (i &= 1, 2, 3, 4). \end{aligned}$$

IV.18 Show that the objective function

$$c'x = x_1 + x_2 + 5x_3 + x_4 + 2x_5$$

has no maximum under the constraints

$$\begin{aligned} x_1 + x_3 - x_4 + x_5 &= 3 \\ x_2 + 4x_3 + x_4 - 3x_5 &= 5 \\ x_i \geq 0 \quad (i &= 1, 2, 3, 4, 5)\ . \end{aligned}$$

IV.19 Determine the value of the parameter γ for which

$$\max c'x = \max (x_1 + \gamma x_2 + \gamma x_3) = 6$$

under the constraints

$$\begin{aligned} x_2 + x_3 &= 2 \\ x_1 + x_2 + (\gamma + 1)x_3 &= \gamma + 2 \\ x_i \geq 0 \quad (i &= 1, 2, 3)\ . \end{aligned}$$

CHAPTER FIVE

THE LINEAR AND THE EUCLIDEAN SPACE

17. The Linear Space

The reader knows the basic operational laws governing the algebraic operations in the set $\mathcal{R}$ of real numbers. In Sections 1, 5, 8, and 9 we met the algebra of vectors in ordinary space, of complex numbers, of matrices, and of ordered n-tuples, respectively. The operational laws in these sets have a number of common features. If we single out some of these common features and disregard the particular sets, the axiomatic introduction of some abstract algebraic structures becomes possible. The study of these abstract structures gives a deeper insight into the properties of the sets mentioned above and makes easier investigation into some further important sets. In this Section the concepts of a 'group', a 'ring', a 'field', and a 'linear space' are introduced.

DEFINITION 17.1. The set $\mathcal{G} = \{a, b, \dots\}$ is called a *group* if an operation denoted by $\circ$ is defined among its elements with the following properties:

a) if $a, b \in \mathcal{G}$, then $a \circ b \in \mathcal{G}$;

b) for all $a, b, c \in \mathcal{G}$
$(a \circ b) \circ c = a \circ (b \circ c)$, (associativity).

c) there exists an $n \in \mathcal{G}$ such that for all $a \in \mathcal{G}$

$$n \circ a = a \circ n = a;$$

n is called the *neutral element* of $\mathcal{G}$;

d) for all $a \in \mathcal{G}$ there is an $a^{-1} \in \mathcal{G}$ such that

$$a \circ a^{-1} = a^{-1} \circ a = n;$$

a^{-1} is called the *inverse* (or *opposite*) of the element a.

DEFINITION 17.2. If $\mathcal{G}$ is a group and, for all $a, b \in G$, $a \circ b = b \circ a$ (commutativity) then $\mathcal{G}$ is called a *commutative* (or *Abelian*) *group*.

The group operation is called either *addition* or *multiplication* and $+$ or $\cdot$ is used instead of $\circ$. In the first case $\mathcal{G}$ is said to be an *additive*, in the second a *multiplicative* group. The group operation is called addition usually only if the group is commutative. In an additive group the neutral element is called *zero* (0), the inverse of an element a is denoted by $-a$.

Note that an operation in which two elements of a set take part, i.e. a so-called binary operation, is actually a function. This function maps the Cartesian product of the set with itself into the set. Thus in the previous definition the group operation is a function $f : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, making to correspond to each ordered pair (a, b) where $a, b \in \mathcal{G}$, the element $a \circ b = f(a, b)$.

Example 17.1. The set of integers forms a commutative group with respect to the operation of common addition. The neutral element is the zero, and the inverse of the integer k is the integer $-k$.

Example 17.2. The regular matrices of order n $(n \geq 1)$ form a non-commutative group with respect to multiplication defined between them (see Definition 8.10 and Example 8.3). According to Theorem 10.16 the product of two regular matrices is again a regular matrix of the same order. According to (8.19) matrix multiplication is associative. The neutral element is the nth-order unit matrix and the inverse of the regular matrix A is A^{-1} (see Theorem 11.2).

It is easy to see that the neutral element n of the group $\mathcal{G}$ is unique Were there an $m \in \mathcal{G}$ such that, say, $m \circ a = a$ for all $a \in \mathcal{G}$, then $m = m \circ n = n$ would follow (we used first the property of n and then the assumption on m). It is also clear that the inverse a^{-1} of an arbitrary element a of the group $\mathcal{G}$ is uniquely determined. If for some $b \in \mathcal{G}$ the equation $b \circ a = n$ (say) held, then multiplying both sides by a^{-1} from the right would give $(b \circ a) \circ a^{-1} = n \circ a^{-1}$, $b \circ (a \circ a^{-1}) = a^{-1}$, $b \circ n = a^{-1}$, $b = a^{-1}$.

Definition 17.3. The set $\mathcal{H} = \{a, b, \ldots\}$ is called a *ring* if two operations, addition $(+)$ and multiplication $(\cdot)$, are defined among its elements with the following properties:

a) $\mathcal{H}$ is a commutative group with respect to addition (the neutral element of this group is called the *zero* of the ring and is denoted by 0);
b) if $a, b \in \mathcal{H}$ then $a \cdot b \in \mathcal{H}$;
c) for all $a, b, c \in \mathcal{H}$
$(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (multiplication is associative);
d) for all $a, b, c \in \mathcal{H}$
$(a + b) \cdot c = a \cdot c + b \cdot c$ and
$c \cdot (a + b) = c \cdot a + c \cdot b$ (multiplication is distributive over addition).

Definition 17.4. If $\mathcal{F}$ is a ring and the set $\mathcal{F} \setminus \{0\}$, consisting of all the elements of $\mathcal{F}$ except the zero, is a commutative group with respect to the multiplication defined in $\mathcal{F}$, then $\mathcal{F}$ is called a *field*; the neutral element of the multiplicative group $\mathcal{F} \setminus \{0\}$ is called the *unit (element)* of $\mathcal{F}$.

Example 17.3. The set $\mathcal{R}$ of real numbers is a field with respect to the operations of common addition and multiplication.

EXAMPLE 17.4. The set $\mathcal{C}$ of complex numbers is a field with respect to the addition and multiplication defined in this set (the multiplicative inverse of a non-zero element is its reciprocal, see Theorem 5.1).

EXAMPLE 17.5. The set of nth-order regular matrices is neither a ring nor a field (with respect to the addition and multiplication defined in Section 10) since, obviously, the sum of two regular matrices may be singular, i.e. may not belong to the same set.

EXAMPLE 17.6. The set of nth-order matrices is a ring (with respect to the addition and multiplication defined in Section 10). The zero is the nth-order zero matrix. This set is not a field, since multiplication is not commutative (neither does every non-zero element of this set have a multiplicative inverse).

DEFINITION 17.5. Let $\mathcal{F} = \{a, b, \ldots\}$ be a field and $\mathcal{L} = \{\mathbf{x}, \mathbf{y}, \ldots\}$ a commutative additive group; we say that $\mathcal{L}$ is a *linear space (vector space) over the field* $\mathcal{F}$ if the product of each element in $\mathcal{F}$ with any element in $\mathcal{L}$ is defined and has the following properties:

a) for all $a \in \mathcal{F}$ and $\mathbf{x} \in \mathcal{L}$
$a\mathbf{x} = \mathbf{x}a \in \mathcal{L}$;
b) if e is the unit of the field $\mathcal{F}$ then for all $\mathbf{x} \in \mathcal{L}$ we have $e\mathbf{x} = \mathbf{x}$;
c) for all $a, b \in \mathcal{F}$ and $\mathbf{x} \in \mathcal{L}$
$a(b\mathbf{x}) = (ab)\mathbf{x}$ and
d) $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$;
e) for all $a \in \mathcal{F}$ and $\mathbf{x}, \mathbf{y} \in \mathcal{L}$
$a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$.

In this book the field $\mathcal{F}$ denotes either the field $\mathcal{R}$ of real numbers or the field $\mathcal{C}$ of complex numbers. The elements of the linear space $\mathcal{L}$ are called *vectors*.

DEFINITION 17.6. Let $\mathcal{L}$ be a linear space over the field $\mathcal{F}$ ($\mathcal{R}$ or $\mathcal{C}$); we say that the vectors $\mathbf{x}_k \in \mathcal{L}$ $(k = 1, 2, \ldots, m)$ are *linearly dependent* if there are numbers $\lambda_k \in \mathcal{F}$ $(k = 1, 2, \ldots, m)$ such that

$$\sum_{k=1}^{m} |\lambda_k| > 0 \quad \text{and} \quad \sum_{k=1}^{m} \lambda_k \mathbf{x}_k = \mathbf{0}$$

where $\mathbf{0}$ is the neutral element of the additive group $\mathcal{L}$ (the *zero vector*; cf. Definition 9.1).

DEFINITION 17.7. Let $\mathcal{L}$ be a linear space over the field $\mathcal{F}$; we say that the vectors $\mathbf{x}_k \in \mathcal{L}$ $(k = 1, 2, \ldots, m)$ are *linearly independent* if they are not *linearly dependent*, i.e. if the equality

$$\sum_{k=1}^{m} \lambda_k \mathbf{x}_k = \mathbf{0}$$

implies $\lambda_k = 0$ $(k = 1, 2, \ldots, m)$ (cf. Definition 9.2).

Definition 17.5 implies that all the formulae corresponding to formulae (9.1)–(9.8) are valid in the space $\mathfrak{Q}^n$ of n-dimensional column vectors (or in $\mathfrak{R}^n$ or in $\mathfrak{S}^n$). Some of these formulae are explicitly stated in Definition 17.5 and the rest can easily be deduced. Further simple natural properties are also valid in a linear space over a field $\mathfrak{F}$. For instance, $0\mathbf{x} = \mathbf{0}$ for all $\mathbf{x} \in \mathfrak{L}$, since by (b) and (d) of Definition 17.5 we have $\mathbf{x} = e\mathbf{x} = (0 + e)\mathbf{x} = 0\mathbf{x} + \mathbf{x}$. Similarly, for all $\lambda \in \mathfrak{F}$ we have $\lambda\mathbf{0} = \mathbf{0}$, since if $\lambda = 0$ then this follows from the previous property, and if $\lambda \neq 0$ then by (c) and (d) $\lambda\mathbf{0} + \mathbf{x} = \lambda\mathbf{0} + \lambda\left(\frac{1}{\lambda}\mathbf{x}\right) = \lambda\left(\mathbf{0} + \frac{1}{\lambda}\mathbf{x}\right) = \lambda\left(\frac{1}{\lambda}\mathbf{x}\right) = \mathbf{x}$ for all $\mathbf{x} \in \mathfrak{L}$.

Theorems analogous to Theorem 9.1, Theorem 9.2, and Corollary 9.3 are valid for vectors in an arbitrary linear space and the proofs are identical in method to those of the theorems quoted. Thus these theorems are not restated here; we refer to the theorems quoted, reading ‘column vector’ and $\mathfrak{Q}^n$ as ‘vector’ and $\mathfrak{L}$, respectively.

Definition 17.8. If there exists a positive integer n such that, in the linear space $\mathfrak{L}$, n linearly independent vectors can be found, but any set of $n + 1$ vectors in $\mathfrak{L}$ is linearly dependent, then $\mathfrak{L}$ is said to be an *n-dimensional linear space* and denoted by $\mathfrak{L}^n$.

Example 17.7. The set $\mathfrak{C}$ of complex numbers is a commutative group with respect to the addition defined in it. It can be seen from (5.9)–(5.11) that $\mathfrak{C}$ is a linear space over the field $\mathfrak{R}$ and, taking into account what has been said in the paragraph preceding (5.23), $\mathfrak{C}$ is two-dimensional.

Example 17.8. The set $\mathfrak{Q}^n$ of the n-dimensional column vectors (the set $\mathfrak{S}^n$ of the n-dimensional row vectors, the set $\mathfrak{R}^n$ or $\mathfrak{C}^n$ of ordered n-tuples) is an n-dimensional linear space over the real or complex field (see Theorem 9.4 and Corollary 9.8).

Example 17.9. Let $a < b$ be given real numbers and consider the set $C^0_{[a,b]}$ of real functions continuous in the interval $[a, b]$. The sum of two continuous functions and a real multiple of a continuous function is again a continuous function. The identically zero function belongs to the set. The operations in question obviously satisfy the requirements of Definition 17.5, and thus $C^0_{[a,b]}$ is a linear space over the field $\mathfrak{R}$ (the identically zero function is its zero). We show that $C^0_{[a,b]}$ is an infinite dimensional linear space, that is, for arbitrary large positive integer $n + 1$ one can find $n + 1$ linearly independent vectors (functions) in $C^0_{[a,b]}$. If $n + 1$ is given let us choose a regular square matrix $Y = [y_{ik}]$ of order $n + 1$ and let $a = x_0 < x_1 < x_2 \ldots < x_i < \ldots < x_n = b$. For $k = 0, 1, 2, \ldots, n$ denote by f_k the continuous function ($f_k \in C^0_{[a,b]}$) whose graph is the polygon connecting the points (x_i, y_{ik}) $(i = 0, 1, 2, \ldots, n)$ in the xy plane, i.e. $f_k(x_i) = y_{ik}$ and

$$f_k(x) = y_{i-1,k} + \frac{y_{ik} - y_{i-1,k}}{x_i - x_{i-1}}(x - x_{i-1}), \qquad x \in [x_{i-1}, x_i]$$
$$(i = 1, 2, \ldots, n).$$

We shall see that the functions $f_0, f_1, \ldots, f_n$ are linearly independent. Assume that, for some real numbers, λ_k $(k = 0, 1, \ldots, n)$

$$\lambda_0 f_0 + \lambda_1 f_1 + \ldots + \lambda_n f_n = 0$$

where 0 on the right-hand side denotes the zero function. This equation must hold for the values of the functions at every $x \in [a, b]$, in particular at the points x_i $(i = 0, 1, \ldots, n)$; thus

$$\sum_{k=0}^{n} y_{ik} \lambda_k = 0, \qquad (i = 0, 1, 2, \ldots, n).$$

This is a homogeneous linear system of equations for $\lambda_0, \lambda_1, \ldots, \lambda_n$. Since its determinant is non-zero, $\det Y \neq 0$, the only solution of the system is the trivial one (see Corollary 15.2), i.e. $\lambda_k = 0$ $(k = 0, 1, \ldots, n)$, and this proves our statement.

18. The Euclidean Space

Multiplication has not been defined for the elements (vectors) of a linear space; neither have we talked about the magnitude of the vectors in an abstract linear space. We reach a more complete generalization of ordinary vector algebra if the scalar product of vectors is introduced. Since the vectors are now the elements of an abstract linear space and cannot be represented, in general, by straight-line segments in ordinary space, *a priori* they have no length and they form no angles with each other. Thus the scalar product of vectors cannot be defined according to Definition 2.1. Instead, we look at the most important properties (2.3), (2.4), (2.5), and (2.7) of the scalar product of (intuitively introduced) vectors in ordinary space and define the scalar product of abstract vectors by postulating these properties.

Definition 18.1. Let $\mathfrak{E}$ be a linear space over the field $\mathfrak{R}$ of real numbers; we say that a *scalar product* has been defined in $\mathfrak{E}$ if a real number is made to correspond to each ordered pair $(\mathbf{x}, \mathbf{y})$ of vectors taken from $\mathfrak{E}$–this is called the scalar product of the two vectors, and is denoted by $\mathbf{x} \cdot \mathbf{y}$–so that for all $\mathbf{x}, \mathbf{y}, \mathbf{x}_1, \mathbf{x}_2 \in \mathfrak{E}$, and $\lambda \in \mathfrak{R}$ we have

a) $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$;
b) $(\lambda \mathbf{x}) \cdot \mathbf{y} = \lambda(\mathbf{x} \cdot \mathbf{y})$;
c) $(\mathbf{x}_1 + \mathbf{x}_2) \cdot \mathbf{y} = \mathbf{x}_1 \cdot \mathbf{y} + \mathbf{x}_2 \cdot \mathbf{y}$;
d) $\mathbf{x} \cdot \mathbf{x} \geq 0$ and $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

The linear space $\mathcal{E}$ with a scalar product is called a *real Euclidean space* and is denoted by $\mathcal{E}(\mathcal{R})$.

The introduction of a scalar product in the linear space $\mathcal{E}$ according to Definition 18.1 means that a functional (a real-valued function) is defined on the Cartesian product of $\mathcal{E}$ with itself which maps this Cartesian product $\mathcal{E}\times\mathcal{E}$ into $\mathcal{R}$ and has some prescribed properties. If this functional is denoted by $f : \mathcal{E}\times\mathcal{E} \to \mathcal{R}$, we have $f(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{E}$. Property a) in Definition 18.1 means that this functional is *symmetric* (in its two arguments). Properties b) and c) together mean that f is linear in each of its arguments, i.e. it is *bilinear*. Property d) means that our functional is *positive definite*.
It is worth introducing the notion of the complex Euclidean space based on a linear space over the complex field, because this makes the situation in some respect simpler than it is in the real Euclidean space (see Section 22).

DEFINITION 18.2. Let $\mathcal{E}$ be a linear space over the field $\mathcal{C}$ of complex numbers; we say that a *scalar product* has been defined in $\mathcal{E}$ if a complex number is made to correspond to each ordered pair $(\mathbf{x}, \mathbf{y})$ of vectors taken from $\mathcal{E}$ — this is called the scalar product of the two vectors, and is denoted by $\mathbf{x} \cdot \mathbf{y}$ — so that for all $\mathbf{x}, \mathbf{y}, \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{E}$, and $\lambda \in \mathcal{C}$ we have

a) $\mathbf{x} \cdot \mathbf{y} = \overline{\mathbf{y} \cdot \mathbf{x}}$ (conjugate symmetry);
b) $(\lambda\mathbf{x}) \cdot \mathbf{y} = \lambda(\mathbf{x} \cdot \mathbf{y})$ and
$\mathbf{x} \cdot (\lambda\mathbf{y}) = \bar{\lambda}(\mathbf{x} \cdot \mathbf{y})$;
c) $(\mathbf{x}_1 + \mathbf{x}_2) \cdot \mathbf{y} = \mathbf{x}_1 \cdot \mathbf{y} + \mathbf{x}_2 \cdot \mathbf{y}$ and
$\mathbf{y} \cdot (\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{y} \cdot \mathbf{x}_1 + \mathbf{y} \cdot \mathbf{x}_2$;
d) $\mathbf{x} \cdot \mathbf{x}$ is real, $\mathbf{x} \cdot \mathbf{x} \geq 0$ and $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

The linear space $\mathcal{E}$ with a scalar product is called a *complex Euclidean space* and denoted by $\mathcal{E}(\mathcal{C})$.

If the scalar product has been introduced into an n-dimensional linear space over the real or complex field according to Definition 18.1 or 18.2, then the space is called an *n-dimensional Euclidean space* and denoted by $\mathcal{E}^n(\mathcal{R})$, or $\mathcal{E}^n(\mathcal{C})$.

DEFINITION 18.3. If $\mathcal{E}$ is a real or complex Euclidean space then the non-negative number

$$||\mathbf{x}|| = \sqrt{\mathbf{x} \cdot \mathbf{x}}, \qquad \mathbf{x} \in \mathcal{E}$$

is called the *norm* of the vector $\mathbf{x}$.

The scalar product of the vectors $\mathbf{x}$ and $\mathbf{y}$ of a Euclidean space is often denoted by $(\mathbf{x}, \mathbf{y})$. We have not applied this notation in the definitions above because we use the same notation for the ordered pair of the two vectors.

Some of the following examples show that some structures defined in earlier chapters of this book satisfy the requirements of Definition 18.1 or 18.2, i.e. they are Euclidean spaces in the abstract sense.

EXAMPLE 18.1. Obviously the vectors of the ordinary space (introduced intuitively in Chapter One) form a three-dimensional real Euclidean space where the norm coincides with the length of a vector.

EXAMPLE 18.2. Let $\mathcal{Q}^n(\mathcal{R})$ denote the n-dimensional linear space of the n-dimensional column vectors with real elements. We define the scalar product $x \cdot y$ of the column vectors $x = \text{col}\,[x_1, \ldots, x_n]$ and $y = \text{col}\,[y_1, \ldots, y_n]$ by

$$x \cdot y = x' y = \sum_{i=1}^{n} x_i y_i \,. \tag{18.1}$$

It is easy to see that (18.1) satisfies the requirements of Definition 18.1 and thus $\mathcal{Q}^n(\mathcal{R})$ becomes an n-dimensional real Euclidean space. The norm of the vector **x** is by Definition 18.3

$$|| x || = (x_1^2 + x_2^2 + \ldots + x_n^2)^{\frac{1}{2}} \,. \tag{18.2}$$

Scalar product can analogously be introduced into the space $\mathcal{S}^n(\mathcal{R})$ of the n-dimensional real row vectors or into the space $\mathcal{R}^n$ of the ordered n-tuples of real numbers.

EXAMPLE 18.3. Let $\mathcal{Q}^n(\mathcal{C})$ denote the n-dimensional linear space of the n-dimensional column vectors with complex elements. We define the scalar product $x \cdot y$ of the column vectors $x = \text{col}\,[x_1, \ldots, x_n]$ and $y = \text{col}\,[y_1, \ldots, y_n]$ by

$$x \cdot y = x' \overline{y} = y^* x = \sum_{i=1}^{n} x_i \overline{y}_i \tag{18.3}$$

(see Definition 8.4). It is easy to see that (18.3) satisfies the requirements of Definition 18.2 and thus $\mathcal{Q}^n(\mathcal{C})$ becomes an n-dimensional complex Euclidean space. The norm of the vector x is by Definition 18.3

$$|| x || = (x_1 \overline{x}_1 + x_2 \overline{x}_2 + \ldots + x_n \overline{x}_n)^{\frac{1}{2}}$$

$$= (| x_1 |^2 + | x_2 |^2 + \ldots + | x_n |^2)^{\frac{1}{2}} \,. \tag{18.4}$$

Scalar product can analogously be introduced into the space $\mathcal{S}^n(\mathcal{C})$ of the complex row vectors and into the space $\mathcal{C}^n$ of the ordered n-tuples of complex numbers.

EXAMPLE 18.4. Let a and b be given real numbers ($a < b$), and consider the set $C^0_{[a,b]}$ of real functions continuous in the closed interval $[a, b]$. By Example 17.9 $C^0_{[a,b]}$ is a linear space over $\mathcal{R}$. We may define a scalar product in this space by

$$f \cdot g = \int_a^b f(x)\, g(x)\, dx, \quad f, g \in C^0_{[a,b]} \,. \tag{18.5}$$

It follows from basic properties of the definite integral that (18.5) satisfies all requirements in Definition 18.1. In particular, since f is continuous,

$$f \cdot f = \int_a^b f^2(x)\, dx = 0$$

implies $f(x) \equiv 0$ (requirement d) in the definition). Thus $C^0_{[a,b]}$ becomes a Euclidean space. The norm of a vector f in $C^0_{[a,b]}$ is

$$\| f \| = \Big[\int_a^b f^2(x)\, dx \Big]^{\frac{1}{2}} . \tag{18.6}$$

An inequality analogous to (2.9) is valid in any real Euclidean space.

THEOREM 18.1. *(Cauchy–Buniakovsky–Schwarz inequality).* In a real Euclidean space $\mathcal{E}(\mathfrak{R})$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{E}(\mathfrak{R})$

$$| \mathbf{x} \cdot \mathbf{y} | \leq \| \mathbf{x} \| \, \| \mathbf{y} \| \tag{18.7}$$

and the equality holds in (18.7) if and only if $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent.

Proof. For arbitrary $\lambda \in \mathfrak{R}$ consider the square of the norm of $\mathbf{x} - \lambda \mathbf{y}$ which is a non-negative number. Applying d) c), b), and a) of Definition 18.1 we have

$$\begin{aligned} 0 \leq \| \mathbf{x} - \lambda \mathbf{y} \|^2 &= (\mathbf{x} - \lambda \mathbf{y}) \cdot (\mathbf{x} - \lambda \mathbf{y}) \\ &= \| \mathbf{x} \|^2 - \lambda(\mathbf{y} \cdot \mathbf{x}) - \lambda(\mathbf{x} \cdot \mathbf{y}) + \lambda^2 \| \mathbf{y} \|^2 \\ &= \| \mathbf{y} \|^2 \lambda^2 - 2(\mathbf{x} \cdot \mathbf{y})\lambda + \| \mathbf{x} \|^2 . \end{aligned}$$

The right-hand side of the last inequality is a quadratic polynomial of the real variable λ. Since it never assumes negative values, it has a single real root at most. Hence

$$4(\mathbf{x} \cdot \mathbf{y})^2 - 4 \, \| \mathbf{y} \|^2 \, \| \mathbf{x} \|^2 \leq 0$$

which amounts to (18.7).

If $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent then we may assume without loss of generality that, say, $\mathbf{x} = \lambda \, \mathbf{y}$ for some $\lambda \in \mathfrak{R}$. But then

$$(\mathbf{x} \cdot \mathbf{y})^2 = (\lambda \mathbf{y} \cdot \mathbf{y})^2 = \lambda^2 \, \| \mathbf{y} \|^4 = \| \lambda \mathbf{y} \|^2 \, \| \mathbf{y} \|^2 = \| \mathbf{x} \|^2 \, \| \mathbf{y} \|^2;$$

thus in (18.7) the equality holds. Conversely, assume that in (18.7) we have an equality and consider the scalar product of a linear combination $\lambda_1 \mathbf{x} + \lambda_2 \mathbf{y}$ with $\mathbf{x}$ and $\mathbf{y}$, respectively. Making these scalar products equal to zero we get a homogeneous linear system of equations in the unknowns λ_1, λ_2 :

$$(\lambda_1 \mathbf{x} + \lambda_2 \mathbf{y}) \cdot \mathbf{x} = \lambda_1 \|\mathbf{x}\|^2 + \lambda_2(\mathbf{x} \cdot \mathbf{y}) = 0$$
$$(\lambda_1 \mathbf{x} + \lambda_2 \mathbf{y}) \cdot \mathbf{y} = \lambda_1(\mathbf{x} \cdot \mathbf{y}) + \lambda_2 \|\mathbf{y}\|^2 = 0.$$

Because of our assumption, the determinant of the system is zero and consequently a non-trivial solution (λ_1, λ_2) exists (see Corollary 15.2). Substituting this solution into the system, multiplying the first identity by λ_1; the second by λ_2, and adding, we get

$$\begin{aligned} 0 &= (\lambda_1 \mathbf{x} + \lambda_2 \mathbf{y}) \cdot (\lambda_1 \mathbf{x}) + (\lambda_1 \mathbf{x} + \lambda_2 \mathbf{y}) \cdot (\lambda_2 \mathbf{y}) \\ &= (\lambda_1 \mathbf{x} + \lambda_2 \mathbf{y}) \cdot (\lambda_1 \mathbf{x} + \lambda_2 \mathbf{y}) = \|\lambda_1 \mathbf{x} + \lambda_2 \mathbf{y}\|^2 . \end{aligned}$$

Hence, by d) of Definition 18.1, $\lambda_1 \mathbf{x} + \lambda_2 \mathbf{y} = \mathbf{0}$. Since not both of λ_1 and λ_2 are zero this means that **x** and **y** are linearly dependent. !

The Cauchy–Buniakovsky–Schwarz inequality also holds for complex Euclidean spaces, as can be proved. But in this book we work mostly in a real Euclidean space and so the previous theorem will suffice.

It is worth indicating certain special cases of the Cauchy–Buniakovsky–Schwarz inequality. Taking into account Example 18.2, we have:

COROLLARY 18.2. *(Cauchy's inequality).* For arbitrary n-tuples of real numbers $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \sum_{i=1}^{n} x_i^2 \sum_{k=1}^{n} y_k^2 , \tag{18.8}$$

and the equality holds if and only if one of the n-tuples is a scalar multiple of the other.

Taking into account Example 18.4, we have:

COROLLARY 18.3. *(Buniakovsky's inequality).* If $f, g \in C^0_{[a,b]}$, i.e. f and g are continuous functions in the interval $[a, b]$ then

$$\left(\int_a^b f(x)\, g(x)\, dx\right)^2 \leq \int_a^b f^2(x)\, dx \int_a^b g^2(x)\, dx, \tag{18.9}$$

and the equality holds if and only if one of the functions is a constant multiple of the other.

Another important consequence of the Cauchy–Buniakovsky–Schwarz inequality is that the analogue of the basic triangle inequality (1.7) is also valid in an abstract Euclidean space.

THEOREM 18.4. *(Triangle inequality, Minkowski's inequality).* In a real Euclidean space $\mathscr{E}(\mathscr{R})$ for all $\mathbf{x}, \mathbf{y} \in \mathscr{E}(\mathscr{R})$

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\| . \tag{18.10}$$

Proof. Applying the defining properties of the scalar product and formula (18.7), we have

$$\begin{aligned}||\mathbf{x}+\mathbf{y}||^2 = (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) &= ||\mathbf{x}||^2 + 2(\mathbf{x}\cdot\mathbf{y}) + ||\mathbf{y}||^2\\ &\le ||\mathbf{x}||^2 + 2\,|\mathbf{x}\cdot\mathbf{y}| + ||\mathbf{y}||^2\\ &\le ||\mathbf{x}||^2 + 2\,||\mathbf{x}||\,||\mathbf{y}|| + ||\mathbf{y}||^2\\ &= (||\mathbf{x}|| + ||\mathbf{y}||)^2 .\,!\end{aligned}$$

The Cauchy–Buniakovsky–Schwarz inequality makes it possible to introduce the concept of the 'angle' of two non-zero vectors in an abstract Euclidean space. In fact, if $\mathbf{x}, \mathbf{y} \in \mathscr{E}(\mathfrak{R})$ and $\mathbf{x} \neq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$, then the inequality

$$-1 \le \frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||\,||\mathbf{y}||} \le 1$$

follows from (18.7). Thus there is one and only one (real) angle φ such that $0 \le \varphi \le \pi$ and

$$\cos\varphi = \frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||\,||\mathbf{y}||}. \tag{18.11}$$

This φ can be called the *angle of the vectors* **x** and **y** (cf. (2.1); in ordinary vector algebra the angle of two non-zero vectors is intuitively known or geometrically defined, and the formula corresponding to (18.11) is a simple consequence of the Definition 2.1 of the scalar product).

Formula (18.11) yields a plausible background for defining the orthogonality of vectors in an abstract Euclidean space. The analogue of Theorem 2.2 arises now as a definition.

DEFINITION 18.4. We say that the vectors $\mathbf{x}, \mathbf{y} \in \mathscr{E}(\mathfrak{R})$ are *orthogonal* if $\mathbf{x} \cdot \mathbf{y} = 0$.

It is clear from the definition that the zero vector is orthogonal to every vector.

19. Bases

This Section deals with decomposition of vectors of finite dimensional linear and Euclidean spaces. Decomposition takes place with respect to some fixed set of vectors, the so-called *base vectors*. Definitions and propositions made for a linear space $\mathfrak{L}^n$ are *a fortiori* valid in a Euclidean space $\mathscr{E}^n$. The converse is not true. In the second half of the Section ideas based on the scalar product are introduced. These are valid in (real) Euclidean spaces but, in general, not in linear spaces without a Euclidean structure.

DEFINITION 19.1. We say that the vectors $\mathbf{e}_k$ ($k = 1, 2, \ldots, n$) of the n-dimensional linear space $\mathfrak{L}^n$ form a *base* if they are linearly independent (see Definition 17.7, cf. Definitions 3.1 and 9.3).

Definition 17.8 implies that there is a base in $\mathfrak{L}^n$.

Theorem 19.1. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ form a base in $\mathfrak{L}^n$; then one can find for each $\mathbf{x} \in \mathfrak{L}^n$ one and only one ordered n-tuple of numbers (real or complex according to whether $\mathfrak{L}^n$ is considered over the field $\mathfrak{R}$ or $\mathfrak{C}$) $(x_1, \ldots, x_n)$ such that

$$\mathbf{x} = \sum_{k=1}^{n} x_k \mathbf{e}_k \tag{19.1}$$

(cf. Theorems 1.5, 9.6 and Corollaries 9.9 and 9.10).

Proof. First the existence of a decomposition (19.1), then its uniqueness will be shown. Since by Definition 17.8 any set of $n + 1$ vectors in $\mathfrak{L}^n$ is linearly dependent, there are numbers λ_k $(k = 1, 2, \ldots, n + 1)$ such that $\sum_{k=1}^{n+1} | \lambda_k | > 0$ and

$$\sum_{k=1}^{n} \lambda_k \mathbf{e}_k + \lambda_{n+1} \mathbf{x} = \mathbf{0} .$$

If λ_{n+1} were zero, some of $\lambda_1, \ldots, \lambda_n$ would be different from zero; thus, a non-trivial linear combination of the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ would be the zero vector, contradicting the fact that $\mathbf{e}_1, \ldots, \mathbf{e}_n$ form a base. Hence $\lambda_{n+1} \neq 0$ and

$$\mathbf{x} = \sum_{k=1}^{n} \left(- \frac{\lambda_k}{\lambda_{n+1}} \right) \mathbf{e}_k .$$

We see that (19.1) is valid for some choice of $x_1, \ldots, x_n$. We show that this choice is unique. Assume that, besides (19.1), we also have

$$\mathbf{x} = \sum_{k=1}^{n} y_k \mathbf{e}_k .$$

Subtracting (19.1) from the last equation we get

$$\mathbf{0} = \sum_{k=1}^{n} (y_k - x_k) \mathbf{e}_k .$$

Since $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are linearly independent, $y_k - x_k = 0$ $(k = 1, \ldots \ldots, n)$ follows, i.e. $y_k = x_k$ $(k = 1, \ldots, n)$. !

The coefficients $x_1, x_2, \ldots, x_n$ in the decomposition (19.1) are called the *coordinates of the vector* $\mathbf{x}$ *in the base* $\mathbf{e}_1, \ldots, \mathbf{e}_n$. By the previous theorem there is a one-to-one correspondence between $\mathfrak{L}^n$ and the set $\mathfrak{R}^n$ or $\mathfrak{C}^n$ (according to whether $\mathfrak{L}^n$ is a linear space over $\mathfrak{R}$ or over $\mathfrak{C}$), provided that a base has been fixed in $\mathfrak{L}^n$. This correspondence is made by associating with each $\mathbf{x} \in \mathfrak{L}^n$ the ordered n-tuple of its coordinates $(x_1, \ldots, x_n)$. Clearly, different vectors of $\mathfrak{L}^n$ have different coordinate n-tuples and for arbitrarily given $y_1, \ldots, y_n$ the vector $\sum_{k=1}^{n} y_k \mathbf{e}_k$ is in $\mathfrak{L}^n$. The zero vector $\mathbf{0}$ of $\mathfrak{L}^n$ is associated with $(0, 0, \ldots, 0)$.

We may write the ordered n-tuples in the form of column or row vectors and, in what has been said above, the role of $\mathfrak{R}^n$ ($\mathfrak{C}^n$) can be played by the set $\mathfrak{Q}^n$ of n-dimensional column vectors or by the set $\mathfrak{S}^n$ of n-dimensional row vectors. For convenience, the coordinate n-tuples of the vector $\mathbf{x}$ are usually represented by the column vector $x = \operatorname{col}[x_1, x_2, \ldots, x_n]$. In the given base the vector $\mathbf{x} \in \mathfrak{L}^n$ is represented by the column vector $x \in \mathfrak{Q}^n$ which will be denoted by

$$\mathbf{x} \simeq x .$$

Let the base $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be fixed in $\mathfrak{L}^n$, $\mathbf{x} \in \mathfrak{L}^n$, $\mathbf{y} \in \mathfrak{L}^n$, $\mathbf{x} \simeq x$, $\mathbf{y} \simeq y$, and λ an arbitrary scalar number; then, clearly,

$$\mathbf{x} + \mathbf{y} \simeq x + y, \ \lambda \mathbf{x} \simeq \lambda x . \tag{19.2}$$

Since linear dependence and independence have similarly been defined in $\mathfrak{L}^n$ and in $\mathfrak{Q}^n$ (cf. Definitions 17.6 and 17.7 with Definitions 9.1 and 9.2) it is clear that if in a fixed base $\mathbf{x}_k \simeq x_{(k)}$ ($k = 1, 2, \ldots, m$) then the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_m$ are linearly dependent (independent) if and only if the column vectors $x_{(1)}, \ldots, x_{(m)}$ are linearly dependent (independent).

We turn now to the representation of vectors in an n-dimensional real Euclidean space $\mathfrak{E}^n(\mathfrak{R})$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be a base in $\mathfrak{E}^n(\mathfrak{R})$ and denote the scalar product of the ith and the kth base vectors by g_{ik}, i.e. set

$$g_{ik} = \mathbf{e}_i \cdot \mathbf{e}_k, \quad (i, k = 1, 2, \ldots, n). \tag{19.3}$$

The symmetry $\mathbf{e}_i \cdot \mathbf{e}_k = \mathbf{e}_k \cdot \mathbf{e}_i$ of the scalar product (a) of Definition 18.1) implies that

$$g_{ik} = g_{ki}, \quad (i, k = 1, 2, \ldots, n). \tag{19.4}$$

The scalar product of the vectors $\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{e}_i$ and $\mathbf{y} = \sum_{k=1}^{n} y_k \mathbf{e}_k$ of $\mathfrak{E}^n(\mathfrak{R})$ is (applying a), b) and c) of Definition 18.1)

$$\mathbf{x} \cdot \mathbf{y} = \left(\sum_{i=1}^{n} x_i \mathbf{e}_i\right) \cdot \left(\sum_{k=1}^{n} y_k \mathbf{e}_k\right) = \sum_{i=1}^{n} \sum_{k=1}^{n} x_i y_k (\mathbf{e}_i \cdot \mathbf{e}_k)$$

$$= \sum_{i,k=1}^{n} g_{ik} x_i y_k . \tag{19.5}$$

The square of the norm of the vector $\|\mathbf{x}\|$ is

$$\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x} = \sum_{i,k=1}^{n} g_{ik} x_i x_k . \tag{19.6}$$

Introducing the quadratic matrix $G = [g_{ik}]$ of order n (which is symmetric because of (19.4)) and applying matrix algebraic notation, we can put (19.5) and (19.6) into the following forms:

$$\mathbf{x} \cdot \mathbf{y} = x' G y \tag{19.7}$$

and

$$\| \mathbf{x} \|^2 = x' G x \tag{19.8}$$

(see Definition 8.3 and formulae (8.18) and (8.14)).

It is particularly easy to determine the scalar product and the norm of vectors if the base vectors are orthogonal to each other and they are unit vectors (their norms are equal to one).

DEFINITION 19.2. A base $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of $\mathcal{E}^n(\mathfrak{R})$ is said to be *orthogonal* if any two vectors of the base are orthogonal, i.e. if

$$\mathbf{e}_i \cdot \mathbf{e}_k = 0 \quad (i \neq k;\ i, k = 1, 2, \ldots, n);$$

if in addition the vectors of the base are unit vectors, that is, if

$$\mathbf{e}_i \cdot \mathbf{e}_k = \delta_{ik} \quad (i, k = 1, 2, \ldots, n) \tag{19.9}$$

where δ_{ik} is the Kronecker symbol (8.3), then the base is said to be *orthonormal* (cf. Definition 3.2).

In an orthonormal base of the space $\mathcal{E}^n(\mathfrak{R})$ the coordinates of a vector have a very simple meaning. If $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is an orthonormal base in $\mathcal{E}^n(\mathfrak{R})$, $\mathbf{x} \in \mathcal{E}^n(\mathfrak{R})$ and (19.1) holds, then taking the scalar product of both sides of (19.1) with the base vector $\mathbf{e}_i$ we get

$$\mathbf{x} \cdot \mathbf{e}_i = \sum_{k=1}^{n} x_k (\mathbf{e}_k \cdot \mathbf{e}_i)$$

or, taking (19.9) into account,

$$x_i = \mathbf{x} \cdot \mathbf{e}_i \quad (i = 1, 2, \ldots, n), \tag{19.10}$$

(cf. (3.7)). In the same orthonormal base (19.3) is transformed into (19.9). Thus, if $\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{e}_i$ and $\mathbf{y} = \sum_{k=1}^{n} y_k \mathbf{e}_k$ then applying (19.5) we get for the scalar product

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i,k=1}^{n} \delta_{ik} x_i y_k = \sum_{k=1}^{n} x_k y_k = x' y \tag{19.11}$$

where $x' = [x_1, \ldots, x_n]$ and $y = \operatorname{col}[y_1, \ldots, y_n]$. We get similarly from (19.6) that

$$\| \mathbf{x} \|^2 = \sum_{k=1}^{n} x_k^2 = x' x \tag{19.12}$$

(cf. (3.11) and (3.12)).

However, we do not yet know whether orthonormal bases exist at all in $\mathcal{E}^n(\mathfrak{R})$. To prove that orthonormal bases exist in a real Euclidean space requires some preparation. First of all the concept of subspace is needed.

DEFINITION 19.3. A subset $\mathfrak{L}^m$ of the linear space $\mathfrak{L}^n$ is called an *m-dimensional subspace* $(0 \leq m \leq n)$ if it is an *m-dimensional* linear space (with respect to the operations defined in $\mathfrak{L}^n$ and, as a consequence, among the elements of $\mathfrak{L}^m$).

This definition is analogous to Definition 9.4; it is shorter because the requirement of being an m-dimensional linear space contains everything that had explicitly been put into Definition 9.4. The important thing is, of course, that taking an arbitrary (finite) number of elements of $\mathfrak{L}^m$, their arbitrary linear combination should belong to $\mathfrak{L}^m$ too. The analogues of Theorems 9.11, 9.12 and Corollaries 9.13, 9.14 are valid and their proofs apply without change. Only the analogue of Theorem 9.12 is recalled as an example.

THEOREM 19.2. If $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \in \mathfrak{L}^n$ are linearly independent then the subset of $\mathfrak{L}^n$ spanned by $\mathbf{a}_1, \ldots, \mathbf{a}_m$ is an *m-dimensional* subspace of $\mathfrak{L}^n$ (and $\mathbf{a}_1, \ldots, \mathbf{a}_m$ is a base of this subspace).

Proof. A repetition of the proof of Theorem 9.12.!

We return now to the n-dimensional real Euclidean space $\mathcal{E}^n(\mathfrak{R})$. Obviously a subspace of $\mathcal{E}^n(\mathfrak{R})$ in the sense of Definition 19.3 is again a Euclidean space (it inherits a scalar product from $\mathcal{E}^n(\mathfrak{R})$). Three lemmas will be proved.

LEMMA 19.3. If $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is a base of $\mathcal{E}^n(\mathfrak{R})$ and the vector $\mathbf{u} \in \mathcal{E}^n(\mathfrak{R})$ is orthogonal to each base vector, i.e.

$$\mathbf{u} \cdot \mathbf{e}_k = 0 \quad (k = 1, 2, \ldots, n)$$

then $\mathbf{u} = \mathbf{0}$.

Proof. $\mathbf{u}$ can be expanded in the following form:

$$\mathbf{u} = \sum_{k=1}^{n} u_k \mathbf{e}_k .$$

Multiplying both sides of this equality by $\mathbf{u}$ we get, because of the assumption

$$\|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u} = \sum_{k=1}^{n} u_k(\mathbf{e}_k \cdot \mathbf{u}) = 0 .$$

It follows, applying d) of Definition 18.1, that $\mathbf{u} = \mathbf{0}$.!

LEMMA 19.4. If $\mathbf{u} \in \mathcal{E}^n(\mathfrak{R})$ and $\mathbf{u} \neq \mathbf{0}$ then vectors of $\mathcal{E}^n(\mathfrak{R})$ that are orthogonal to $\mathbf{u}$ form an $(n-1)$-dimensional subspace $\mathcal{E}^{n-1}$ of $\mathcal{E}^n(\mathfrak{R})$.

Proof. It is easy to see that the subset $\mathcal{E}^{n-1}$ of the vectors orthogonal to $\mathbf{u}$ is a subspace. Since if $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{E}^{n-1}$ and $c_1, c_2 \in \mathfrak{R}$ are arbitrarily given then $\mathbf{u} \cdot (c_1\mathbf{x}_1 + c_2\mathbf{x}_2) = c_1(\mathbf{u} \cdot \mathbf{x}_1) + c_2(\mathbf{u} \cdot \mathbf{x}_2) = 0$; hence $(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) \in \mathcal{E}^{n-1}$. It is to be proved that $\mathcal{E}^{n-1}$ is $(n-1)$-dimensional.

Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be a base in $\mathscr{E}^n(\mathscr{R})$, $\mathbf{x} \in \mathscr{E}^{n-1}$ arbitrarily given, and

$$\mathbf{x} = \sum_{k=1}^{n} x_k \, \mathbf{e}_k \, .$$

Since $\mathbf{x} \in \mathscr{E}^{n-1}$, therefore

$$0 = \mathbf{u} \cdot \mathbf{x} = \sum_{k=1}^{n} x_k(\mathbf{u} \cdot \mathbf{e}_k) \, .$$

At least one of the numbers $\mathbf{u} \cdot \mathbf{e}_k$ is non-zero since otherwise by the previous lemma $\mathbf{u} = \mathbf{0}$ would hold, contrary to the assumption. Let, say, $\mathbf{u} \cdot \mathbf{e}_n \neq 0$. Then from the last equality x_n can be expressed,

$$x_n = \sum_{k=1}^{n-1} \lambda_k \, x_k \qquad \text{where} \qquad \lambda_k = -\frac{\mathbf{u} \cdot \mathbf{e}_k}{\mathbf{u} \cdot \mathbf{e}_n}$$
$$(k = 1, 2, \ldots, n-1).$$

Substituting this expression for x_n into the decomposition of the vector $\mathbf{x}$ we get

$$\mathbf{x} = \sum_{k=1}^{n-1} x_k \, \mathbf{e}_k + \left(\sum_{k=1}^{n-1} \lambda_k \, x_k \right) \mathbf{e}_n = \sum_{k=1}^{n-1} x_k(\mathbf{e}_k + \lambda_k \, \mathbf{e}_n) \, .$$

Thus any vector $\mathbf{x}$ of $\mathscr{E}^{n-1}$ is a linear combination of the vectors $\mathbf{e}_k + \lambda_k \, \mathbf{e}_n$ $(k = 1, 2, \ldots, n-1)$. Conversely, if $\mathbf{x}$ is a linear combination of these vectors, then it is orthogonal to $\mathbf{u}$, i.e. it belongs to $\mathscr{E}^{n-1}$. The latter statement follows from the fact that $\mathbf{u}$ is orthogonal to these vectors:

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{e}_k + \lambda_k \, \mathbf{e}_n) &= \mathbf{u} \cdot \mathbf{e}_k + \lambda_k \mathbf{u} \cdot \mathbf{e}_n \\ &= \mathbf{u} \cdot \mathbf{e}_k - \frac{\mathbf{u} \cdot \mathbf{e}_k}{\mathbf{u} \cdot \mathbf{e}_n} \, \mathbf{u} \cdot \mathbf{e}_n = 0 \end{aligned}$$
$$(k = 1, 2, \ldots, n-1).$$

Thus $\mathbf{x} \in \mathscr{E}^{n-1}$ if and only if it is a linear combination of $\mathbf{e}_k + \lambda_k \mathbf{e}_n$ $(k = 1, 2, \ldots, n-1)$, that is, $\mathscr{E}^{n-1}$ is the subspace spanned by these vectors. But these vectors are linearly independent since if

$$\sum_{k=1}^{n-1} \mu_k(\mathbf{e}_k + \lambda_k \, \mathbf{e}_n) = \mathbf{0} \, ,$$

then

$$\sum_{k=1}^{n-1} \mu_k \, \mathbf{e}_k + \left(\sum_{k=1}^{n-1} \mu_k \, \lambda_k \right) \mathbf{e}_n = \mathbf{0} \, ,$$

and taking into account the linear independence of $\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}, \mathbf{e}_n$ we obtain $\mu_1 = \mu_2 = \ldots = \mu_{n-1} = 0$.!

Lemma 19.5. If $\mathbf{e}_k \in \mathcal{E}^n(\mathfrak{R})$ $(k = 1, 2, \ldots, n)$ is an orthonormal system of vectors, that is to say

$$\mathbf{e}_i \cdot \mathbf{e}_k = \delta_{ik} \quad (i, k = 1, 2, \ldots, n),$$

then $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is a base of the space $\mathcal{E}^n(\mathfrak{R})$.

Proof. It is to be shown that $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are linearly independent. Assume that $\sum_{k=1}^{n} \lambda_k \mathbf{e}_k = \mathbf{0}$ and take the scalar product of the left-hand side of this equality with $\mathbf{e}_i$. We get

$$0 = \mathbf{e}_i \cdot \sum_{k=1}^{n} \lambda_k \mathbf{e}_k = \sum_{k=1}^{n} \lambda_k \delta_{ik} = \lambda_i$$

$$(i = 1, 2, \ldots, n),$$

i.e. $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$ follows and this proves our statement. !

Theorem 19.6. In the space $\mathcal{E}^n(\mathfrak{R})$ there is an orthonormal base.

Proof. The proof needs repeated application of Lemma 19.4. Let $\mathbf{e}_1 \in \mathcal{E}^n(\mathfrak{R})$, $\| \mathbf{e}_1 \| = 1$ and denote the subspace of the vectors orthogonal to $\mathbf{e}_1$ by $\mathcal{E}^{n-1}$. Let $\mathbf{e}_2 \in \mathcal{E}^{n-1}$ $\| \mathbf{e}_2 \| = 1$ and denote the subspace of the vectors of $\mathcal{E}^{n-1}$ that are orthogonal to $\mathbf{e}_2$ by $\mathcal{E}^{n-2}$. Obviously, $\mathcal{E}^{n-2}$ being an $(n-2)$-dimensional subspace of the space $\mathcal{E}^{n-1}$ it is also an $(n-2)$-dimensional subspace of $\mathcal{E}^n(\mathfrak{R})$. Let $\mathbf{e}_3 \in \mathcal{E}^{n-2}$, $\| \mathbf{e}_3 \| = 1$. Clearly, $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$, $\mathbf{e}_2 \cdot \mathbf{e}_3 = 0$ and since $\mathbf{e}_3 \in \mathcal{E}^{n-2} \subset \mathcal{E}^{n-1}$, it follows that $\mathbf{e}_3 \cdot \mathbf{e}_1 = 0$. Denoting by $\mathcal{E}^{n-3}$ the subspace of the vectors of $\mathcal{E}^{n-2}$ that are orthogonal to $\mathbf{e}_3$ and repeating this process n times, n vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_n$ are chosen from $\mathcal{E}^n$ in such a way that

$$\mathbf{e}_i \cdot \mathbf{e}_k = \delta_{ik} \quad (i, k = 1, 2, \ldots, n).$$

These vectors form an orthonormal system of vectors and thus by Lemma 19.5 they form an orthonormal base. !

20. Transformation of Bases

In this Section we study what happens to the coordinates of a vector (to the column vector representing the vector) if a base is replaced by another one, i.e. if a transformation of the base takes place.

Consider two bases of the n-dimensional linear space $\mathfrak{L}^n$: $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ and $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \ldots, \tilde{\mathbf{e}}_n$. The first one will be called the 'old' and the second the 'new' base. The base transformation is considered to be given or known if the vectors of the new base are expressed in the old base, i.e. as linear combinations of the vectors of the old base:

$$\tilde{\mathbf{e}}_k = \sum_{i=1}^{n} t_{ik} \mathbf{e}_i \qquad (k = 1, 2, \ldots, n). \tag{20.1}$$

The nth-order quadratic matrix $T = [t_{ik}]$ will be called the *base transformation matrix.* The kth column vector $t_k = \operatorname{col}[t_{1k}, t_{2k}, \ldots, t_{nk}]$ of the matrix T is the column vector representing the vector $\tilde{\mathbf{e}}_k$ in the old base, that is, $\tilde{\mathbf{e}}_k \simeq t_k$. Since the vectors $\tilde{\mathbf{e}}_k$ $(k = 1, 2, \ldots, n)$ are linearly independent, the transformation matrix T is regular det $T \neq 0$.

For an arbitrary vector $\mathbf{x}$ of $\mathfrak{L}^n$ we have the unique decomposition in each base:

$$\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{e}_i\,, \qquad \mathbf{x} = \sum_{k=1}^{n} \tilde{x}_k \tilde{\mathbf{e}}_k\,, \tag{20.2}$$

that is, the column vectors representing $\mathbf{x}$ in the old and in the new base are $\mathbf{x} \simeq x = \operatorname{col}[x_1 \ldots, x_n]$ and $\mathbf{x} \simeq \tilde{x} = \operatorname{col}[\tilde{x}_1, \ldots, \tilde{x}_n]$, respectively. We proceed to find the relation between the column vectors x and $\tilde{x}$. Substitute (20.1) into the second formula of (20.2); interchanging the order of summation we get

$$\mathbf{x} = \sum_{k=1}^{n} \tilde{x}_k \sum_{i=1}^{n} t_{ik}\mathbf{e}_i = \sum_{i,k=1}^{n} \tilde{x}_k t_{ik}\mathbf{e}_i = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} t_{ik}\tilde{x}_k\right)\mathbf{e}_i\,.$$

Since the decomposition of $\mathbf{x}$ in the old base is uniquely determined, comparing the last expression with the first formula of (20.2) we see that the following transformation formula holds:

$$x_i = \sum_{k=1}^{n} t_{ik}\tilde{x}_k \qquad (i = 1, 2, \ldots, n)$$

or in matrix notation

$$x = T\tilde{x}\,. \tag{20.3}$$

As we have seen, T is regular and so has an inverse matrix T^{-1}. Multiplying both sides of (20.3) by T^{-1} from the left we get

$$T^{-1}x = T^{-1}T\tilde{x} = I\tilde{x} = \tilde{x}\,.$$

Thus the column vector $\tilde{x}$ representing the vector $\mathbf{x}$ in the new base is obtained by the transformation formula

$$\tilde{x} = T^{-1}x\,. \tag{20.4}$$

We now consider the important particular case in which the space is the n-dimensional real Euclidean space $\mathcal{E}^n(\mathfrak{R})$ and both the old and the new bases are orthonormal. The replacement of an orthonormal base with another orthonormal base is called an *orthogonal base transformation.* Assume that the bases $\mathbf{e}_1, \ldots, \mathbf{e}_n$ and $\tilde{\mathbf{e}}_1, \ldots, \tilde{\mathbf{e}}_n$ are both orthonormal (see Definition 19.2) and the base transformation is

$$\tilde{\mathbf{e}}_k = \sum_{i=1}^{n} m_{ik}\mathbf{e}_i \qquad (k = 1, 2, \ldots, n)\,.$$

The kth column vector of the matrix $M = [m_{ik}]$ of the orthogonal base transformation will be denoted by m_k ($k = 1, 2, \ldots, n$). In the old base, $\tilde{\mathbf{e}}_k \simeq m_k$. Applying (19.9) and (19.11) to the vectors $\tilde{\mathbf{e}}_k, \tilde{\mathbf{e}}_l$ we get the conditions

$$m_k' m_l = \delta_{kl} \quad (k, l = 1, 2, \ldots, n) \tag{20.5}$$

for the column vectors of the matrix M. Obviously, M is the transformation matrix of an orthogonal base transformation if and only if (20.5) holds for its column vectors. Conditions (20.5) characterize an important class of matrices, the orthogonal matrices.

Definition 20.1. The quadratic matrix M of order n is called an *orthogonal matrix* if

$$M' M = I \tag{20.6}$$

where M' is the transpose of M and I is the unit matrix.

If the column vectors of the matrix M are denoted by $m_1, m_2, \ldots, m_n$ successively then the row vectors of the transpose M' are $m_1', m_2', \ldots, m_n'$. Thus, the defining property (20.6) of an orthogonal matrix is equivalent to (20.5). The latter form of the defining property is sometimes expressed as follows: The column vectors of an orthogonal matrix are unit vectors orthogonal to each other. Further important properties of orthogonal matrices are given below.

If the matrix M is orthogonal then it is regular and

$$\det M = \pm 1, \tag{20.7}$$

since applying Theorem 10.16 and Corollary 10.14 we get from (20.6)

$$1 = \det I = \det M' \det M = (\det M)^2.$$

M is orthogonal if and only if it is regular and its inverse is equal to its transpose:

$$M' = M^{-1}. \tag{20.8}$$

This is an immediate consequence of (20.6) and of the definition of the inverse matrix (Definition 11.1).

If M is orthogonal then its transpose is also orthogonal, since by (20.8)

$$(M')'M' = MM' = MM^{-1} = I. \tag{20.9}$$

The column vectors of M' are the row vectors of M, and so it follows that the row vectors of an orthogonal matrix are also unit vectors orthogonal to each other.

Returning to the problem of orthogonal transformation of bases we have seen that the matrix M of such a transformation is orthogonal. Thus, taking (20.8) into account, the transformation formula (20.4) of the coordinate column vectors assumes the form

$$\tilde{x} = M' x. \tag{20.10}$$

Example 20.1. In the ordinary (two-dimensional Euclidean) plane, consider a Cartesian orthogonal coordinate system x_1, x_2 defined by a point O (the origin) and two orthogonal unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. Rotating this coordinate system around the origin by an angle α we obtain a new Cartesian orthogonal coordinate system $\tilde{x}_1$, $\tilde{x}_2$ with the same origin and with base vectors $\tilde{\mathbf{e}}_1$, $\tilde{\mathbf{e}}_2$; the latter make an angle α with $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively (see Fig. 20.1). Applying (3.8) and simple trigonometric formulae

$$\begin{aligned} \tilde{\mathbf{e}}_1 &= \cos\alpha\, \mathbf{e}_1 + \sin\alpha\, \mathbf{e}_2 \\ \tilde{\mathbf{e}}_2 &= -\sin\alpha\, \mathbf{e}_1 + \cos\alpha\, \mathbf{e}_2. \end{aligned} \tag{20.11}$$

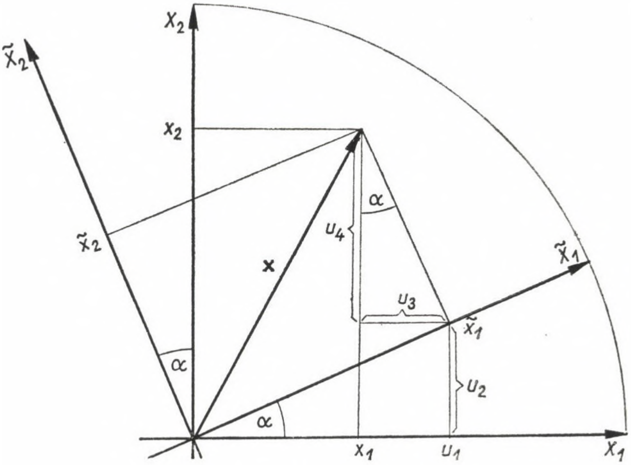


Fig. 20.1

The matrix of this orthogonal base transformation is

$$M = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

The matrix M is indeed orthogonal, and $\det M = 1$. The transformation formula of the coordinate column vectors representing the position vectors, i.e. the points in the plane, is by (20.10)

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{20.12}$$

This result can be deduced by simple trigonometric reasoning if, introducing $u_1 = \tilde{x}_1 \cos\alpha$, $u_2 = \tilde{x}_1 \sin\alpha$, $u_3 = \tilde{x}_2 \sin\alpha$, $u_4 = \tilde{x}_2 \cos\alpha$ we take into account that

$$\begin{aligned} x_1 &= u_1 - u_3 = \tilde{x}_1 \cos\alpha - \tilde{x}_2 \sin\alpha \\ x_2 &= u_2 + u_4 = \tilde{x}_1 \sin\alpha + \tilde{x}_2 \cos\alpha \end{aligned}$$

and express $\tilde{x}_1$ and $\tilde{x}_2$ from this system of equations.

If the rotation of the coordinate system is followed by a reflection in the axis $\tilde{x}_1$ and the new base vectors are denoted by $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ then because $\hat{\mathbf{e}}_1 = \tilde{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2 = -\tilde{\mathbf{e}}_2$ we have

$$\begin{aligned} \tilde{\mathbf{e}}_1 &= \cos\alpha\, \mathbf{e}_1 + \sin\alpha\, \mathbf{e}_2 \\ \hat{\mathbf{e}}_2 &= \sin\alpha\, \mathbf{e}_1 - \cos\alpha\, \mathbf{e}_2\,. \end{aligned} \tag{20.13}$$

The base transformation is still orthogonal, of course, but the determinant of the transformation is equal to -1.

Example 20.2. Having fixed the origin O in the ordinary plane, consider two arbitrarily given orthonormal bases, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\tilde{\mathbf{e}}_1$, $\tilde{\mathbf{e}}_2$, and let the base transformation be

$$\begin{aligned} \tilde{\mathbf{e}}_1 &= m_{11}\, \mathbf{e}_1 + m_{21}\, \mathbf{e}_2 \\ \tilde{\mathbf{e}}_2 &= m_{12}\, \mathbf{e}_1 + m_{22}\, \mathbf{e}_2. \end{aligned}$$

We show that this orthogonal base transformation represents the rotation of one base into the other if

$$\det\,[m_{ik}] = 1$$

and a rotation followed by a reflection (in one of the axes of the new coordinate system) if the value of this determinant is -1. In other words, every base transformation given by an orthogonal matrix of order two represents a rotation or a rotation and a reflection.

Since the matrix $M = [m_{ik}]$ is orthogonal the following equalities hold:

$$m_{11}^2 + m_{21}^2 = 1 \tag{20.14}$$

$$m_{12}^2 + m_{22}^2 = 1 \tag{20.15}$$

$$m_{11}\, m_{12} + m_{21}\, m_{22} = 0\,. \tag{20.16}$$

As a consequence, no column or row of M may consist of zeros only. Further it is easy to see that if M has an element equal to zero then the elements in one diagonal of M are zero and the elements in the other diagonal are equal to plus or minus one. For instance if $m_{21} = 0$ then by (20.14) $m_{11} = \pm 1$, by (20.16) $m_{12} = 0$ and by (20.15) $m_{22} = \pm 1$. Thus either

$$M = \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix} \quad \text{or} \quad M = \begin{bmatrix} \pm 1 & 0 \\ 0 & \mp 1 \end{bmatrix}.$$

In the first case $\det M = 1$ and the transformation represents a rotation by angle zero or π. In the second case $\det M = -1$ and the transformation is a reflection in one of the coordinate axes. If no element of M is zero then, obviously, $|\, m_{11} \,| < 1$ and there is an angle α such that $0 < \alpha < \pi$, $\alpha \neq \pi/2$ and $m_{11} = \cos\alpha$. By (20.14) $m_{21} = \pm \sin\alpha$, by (20.16) $m_{12} = \mp \tan\alpha\, m_{22}$. Substituting these results into (20.15) we get

$$\tan^2\alpha\; m_{22}^2 + m_{22}^2 = 1$$

i.e. $m_{22} = \pm \cos\alpha$ and thus $m_{12} = \mp \sin\alpha$. Collecting these results either

$$M = \begin{bmatrix} \cos\alpha & \mp\sin\alpha \\ \pm\sin\alpha & \cos\alpha \end{bmatrix} \quad \text{or} \quad M = \begin{bmatrix} \cos\alpha & \pm\sin\alpha \\ \pm\sin\alpha & -\cos\alpha \end{bmatrix}.$$

If

$$M = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \quad \text{or} \quad M = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix},$$

then these transformations are identical to those of (20.11) and (20.13), respectively. If

$$M = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \quad \text{or} \quad M = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & -\cos\alpha \end{bmatrix},$$

then with $\beta = -\alpha$ as the angle of rotation the latter transformations reduce to the former ones with β instead of α.

EXAMPLE 20.3. In the ordinary (three-dimensional Euclidean) space, consider a Cartesian orthogonal coordinate system x_1, x_2, x_3 defined by a point O (the origin) and three orthogonal unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ forming a dextral system. Rotating this system around the origin we get a new Cartesian orthogonal coordinate system $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ with the same origin and base vectors $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3$. The base transformation is orthogonal and the new base vectors can be expressed in terms of the old ones, using their direction cosines (see (3.8)):

$$\begin{aligned} \tilde{\mathbf{e}}_1 &= \cos\alpha_1\,\mathbf{e}_1 + \cos\alpha_2\,\mathbf{e}_2 + \cos\alpha_3\,\mathbf{e}_3 \\ \tilde{\mathbf{e}}_2 &= \cos\beta_1\,\mathbf{e}_1 + \cos\beta_2\,\mathbf{e}_2 + \cos\beta_3\,\mathbf{e}_3 \\ \tilde{\mathbf{e}}_3 &= \cos\gamma_1\,\mathbf{e}_1 + \cos\gamma_2\,\mathbf{e}_2 + \cos\gamma_3\,\mathbf{e}_3. \end{aligned}$$

The matrix of the base transformation is

$$M = \begin{bmatrix} \cos\alpha_1 & \cos\beta_1 & \cos\gamma_1 \\ \cos\alpha_2 & \cos\beta_2 & \cos\gamma_2 \\ \cos\alpha_3 & \cos\beta_3 & \cos\gamma_3 \end{bmatrix}. \tag{20.17}$$

$\det M = 1$ because $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3$ have been obtained from $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ by a rotation and so they also form a dextral system (their scalar triple product is plus one). If (x_1, x_2, x_3) is the coordinate triple of a position vector (point) in the old base then the coordinate triple $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ in the new base is obtained according to (20.10), by

$$\begin{aligned} \tilde{x}_1 &= \cos\alpha_1\,x_1 + \cos\alpha_2\,x_2 + \cos\alpha_3\,x_3 \\ \tilde{x}_2 &= \cos\beta_1\,x_1 + \cos\beta_2\,x_2 + \cos\beta_3\,x_3 \\ \tilde{x}_3 &= \cos\gamma_1\,x_1 + \cos\gamma_2\,x_2 + \cos\gamma_3\,x_3\,. \end{aligned}$$

If the rotation discussed above is followed by a reflection in the plane through the origin spanned by $\mathbf{e}_1, \mathbf{e}_2$, say, then we get a new base $\hat{\mathbf{e}}_1 = \tilde{\mathbf{e}}_1, \hat{\mathbf{e}}_2 = \tilde{\mathbf{e}}_2, \hat{\mathbf{e}}_3 = -\tilde{\mathbf{e}}_3$. The matrix of the base transformation is derived from (20.17) by multiplying the last column of the latter by -1. The vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ form a sinistral system and the determinant of the transformation matrix is now equal to -1.

Exercises

V.1 Is the following set a group under addition?
- a) The set of even numbers.
- b) The set of odd numbers.
- c) $\mathfrak{I}_m = \{ l : l = m \cdot k,\ k = 0, \pm 1, \pm 2, \pm \ldots \}$ where m is a fixed natural number.
- d) The set of pure imaginary complex numbers.
- e) The set of n-dimensional row (or column) vectors $\mathcal{S}^n$ (or $\mathcal{Q}^n$).

V.2 Is the following set a group under multiplication ?
- a) The set of integers.
- b) The set of those complex numbers z that satisfy $\mathrm{Re}^2 z + \mathrm{Im}^2 z \neq 0$.
- c) The set of rational numbers, zero excluded.
- d) $\mathcal{G} = \{ c : c = a^k,\ k = 0, \pm 1, \ldots \}$ where $a \neq 0$ is real,
- e) $\mathcal{G} = \{ 1, -1, i, -i \}$, where i is the imaginary unit.
- f) The third roots of unity.
- g) The set of positive real numbers.

V.3 Show that the real functions with common domain [0,1] form a ring under addition and multiplication.

V.4 $\mathcal{H} = \{ h : h = 3^n,\ n = 0, \pm 1, \ldots \}$ the first operation $\circ$ and the second operation $*$ are defined in $\mathcal{H}$ by $3^n \circ 3^m = 3^{n+m}$ and $3^n * 3^m = 3^{n \cdot m}$, respectively. Show that $\mathcal{H}$ is a ring.

V.5 $\mathfrak{I}$ is the set of integers. The first operation is addition. The second operation $*$ is defined by $a * b = 0,\ a, b \in \mathfrak{I}$. Is it a ring ?

V.6 Show that for any fixed positive prime number p the set $\mathfrak{F} = \{x + y\sqrt{p};\ x, y \text{ rational numbers}\}$ is a field under ordinary addition and multiplication.

V.7 Let $\mathcal{V} \subset \mathfrak{L}$, where $\mathfrak{L}$ is a linear space over the field $\mathfrak{F}$. Show that $\mathcal{V}$ is a linear space over the field $\mathfrak{F}$ if and only if, for every $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{V}$ and $\lambda_1, \lambda_2 \in \mathfrak{F}$: $\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 \in \mathcal{V}$.

V.8 Which of the following subsets of $\mathcal{S}^3(\mathfrak{R})$ (the space of three-dimensional real row vectors) is a linear space over $\mathfrak{R}$?
- a) $\mathcal{A} = \{ [a, a, 0] : a \in \mathfrak{R} \}$
- b) $\mathcal{A} = \{ [2a_1, 4a_1, a_2], : a_1, a_2 \in \mathfrak{R} \}$
- c) $\mathcal{A} = \{ [1, a_1, a_2] : a_1\ a_2 \in \mathfrak{R} \}$
- d) $\mathcal{A} = \{ [a_1, a_2, a_3] : a_i \in \mathfrak{R},\ i = 1, 2, 3,\ a_3 = a_1 + a_2 \}$.

V.9 Show that the set $\mathcal{S}^2(\mathfrak{R})$ of two-dimensional real row vectors is not a linear space if addition is defined by

$$[x_1, x_2] + [y_1, y_2] = [x_1 + y_1, 0]$$

and multiplication by a scalar in the ordinary way.

V.10 Show that the set $\mathcal{M}$ of $m \times n$ matrices is an mn-dimensional linear space over $\mathcal{R}$ (or $\mathcal{C}$) under common matrix addition and multiplication by a scalar.

V.11 Which of the following subsets of the set $\mathcal{M}$ of nth-order quadratic matrices is a subspace of $\mathcal{M}$ (see Exercise V.10; $A = [a_{ik}]$)?
a) $\{A \in \mathcal{M} : \det A = 0\}$
b) $\{A \in \mathcal{M} : \det A \neq 0\}$
c) $\{A \in \mathcal{M} : a_{ii} = 0,\ i = 1, 2, \ldots, n\}$
d) $\{A \in \mathcal{M} : a_{ik} = 0,\ \text{if } i \neq k\}$.

V.12 Determine a base in the linear space of matrices of order two.

V.13 Which of the following subsets of $C^0_{[a,\,b]}$ is a subspace (see Example 17.9)?
a) The set of (real) polynomials of even degree.
b) The set of continuous functions that assume the value zero at a finite number of points only.
c) $\{f \in C^0_{[a,\,b]} : f(a) = f(b)\}$.
d) For a fixed positive integer n the set of polynomials of degree n at most.
e) For a fixed positive integer n the set of polynomials of degree n at most with integer coefficients.

V.14 Consider the linear space of second-order real matrices (see Exercise V.10) and let $A = [a_{ik}]$, $B = [b_{ik}]$ be two elements of this space. Is

$$A \cdot B = \begin{vmatrix} a_{11}b_{11} & a_{12}b_{12} \\ a_{21}b_{21} & a_{22}b_{22} \end{vmatrix}$$

an appropriate definition of the scalar product?

V.15 Find the points (x, y) on the plane having the property that the vectors

$$\begin{bmatrix} \sin x \\ \cos x \end{bmatrix}, \qquad \begin{bmatrix} \cos y \\ \sin y \end{bmatrix}$$

of the Euclidean space $\mathcal{Q}^2(\mathcal{R})$ (see Example 18.2) are orthogonal.

V.16 In the subspace of the Euclidean space $C^0_{[0,\,1]}$ (see Example 18.4) consisting of the real polynomials of degree 2 at most (see Exercise V.13) the polynomials $p_0(x) \equiv 1$, $p_1(x) \equiv x$, $p_2(x) \equiv x^2$ form a base.
Find an orthogonal base in this subspace.

V.17 Show that the functions

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{\cos x}{\sqrt{\pi}}, \quad \frac{\sin x}{\sqrt{\pi}}, \quad \ldots \quad \frac{\cos nx}{\sqrt{\pi}}, \quad \frac{\sin nx}{\sqrt{\pi}}, \quad \ldots$$

form an orthonormal system in the Euclidean space $C^0_{[-\pi,\,\pi]}$ (see Example 18.4).

V.18 Given an orthonormal base in $\mathcal{E}^3(\mathcal{R})$, $\mathbf{e}_i$ $(i = 1, 2, 3)$, a new base $\tilde{\mathbf{e}}_i$ $(i = 1, 2, 3)$ is given by

$$\tilde{\mathbf{e}}_1 = \frac{1}{\sqrt{3}}\,\mathbf{e}_1 - \frac{1}{\sqrt{3}}\,\mathbf{e}_2 + \frac{1}{\sqrt{3}}\,\mathbf{e}_3$$

$$\tilde{\mathbf{e}}_2 = \frac{1}{\sqrt{2}}\,\mathbf{e}_1 + \frac{1}{\sqrt{2}}\,\mathbf{e}_2$$

$$\tilde{\mathbf{e}}_3 = -\frac{1}{\sqrt{6}}\,\mathbf{e}_1 + \frac{1}{\sqrt{6}}\,\mathbf{e}_2 + \frac{2}{\sqrt{6}}\,\mathbf{e}_3 .$$

a) Is the new base an orthonormal one?
b) What are the new coordinates of the vector $\mathbf{x} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$?
c) Find $\| \mathbf{x} \|$, carrying through the calculation in both bases.

V.19 In the Euclidean space $\mathcal{Q}^2(\mathfrak{R})$ (see Example 18.2) $\mathbf{e}_1 = \operatorname{col}[1, 0]$, $\mathbf{e}_2 = \operatorname{col}[0, 1]$ is an orthonormal base. Another base is given by

$$\begin{aligned} \tilde{\mathbf{e}}_1 &= 2\mathbf{e}_1 + \mathbf{e}_2 \\ \tilde{\mathbf{e}}_2 &= \mathbf{e}_1 + 2\mathbf{e}_2. \end{aligned}$$

a) Find the new coordinates of the vectors $\mathbf{x}_1 = \operatorname{col}[1, 1]$ and $\mathbf{x}_2 = \operatorname{col}[-1, 2]$.
b) Find the scalar product $\mathbf{x}_1 \cdot \mathbf{x}_2$, carrying through the calculation in both bases.

V.20 Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be a dextral orthonormal base in the ordinary (three-dimensional Euclidean) space and $\tilde{\mathbf{e}}_i$ the reflection of the position vector $\mathbf{e}_i$ in the line passing through the origin with direction vector $\mathbf{v} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ $(i = 1, 2, 3)$. Find the matrix of the base transformation when $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3$ is introduced as a new base.

CHAPTER SIX

LINEAR OPERATORS

21. Operator Algebra

This Chapter deals with linear functions mapping the n-dimensional linear (or Euclidean) space into itself. These functions play an important role in applications such as vector analysis, mechanics, potential theory and electrodynamics, and have important generalizations in functional analysis, in the theory of integral equations, etc.

DEFINITION 21.1. The function $\mathbf{A} : \mathfrak{L}^n \to \mathfrak{L}^n$ mapping the n-dimensional (real or complex) linear space $\mathfrak{L}^n$ into itself, making a vector $\mathbf{y} \in \mathfrak{L}^n$, denoted by $\mathbf{y} = \mathbf{Ax}$, correspond to each $\mathbf{x} \in \mathfrak{L}^n$, is called a *linear operator* if for any $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2 \in \mathfrak{L}^n$ and λ scalar (real or complex number)

a) $\mathbf{A}(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{Ax}_1 + \mathbf{Ax}_2$,
b) $\mathbf{A}(\lambda\mathbf{x}) = \lambda(\mathbf{Ax})$.

Requirements a) and b) in Definition 21.1 are equivalent to the following:

for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathfrak{L}^n$ and scalars λ_1, λ_2

$$\mathbf{A}(\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2) = \lambda_1(\mathbf{Ax}_1) + \lambda_2(\mathbf{Ax}_2) . \tag{21.1}$$

Two linear operators are said to be *equal* if their respective values are equal at every $\mathbf{x} \in \mathfrak{L}^n$, i.e. if they map the same vector into the same vector. The function mapping $\mathfrak{L}^n$ into $\mathfrak{L}^n$, and making the zero vector correspond to each $\mathbf{x} \in \mathfrak{L}^n$, is called the *zero operator* and is denoted by $\mathbf{0}$. Thus for all $\mathbf{x} \in \mathfrak{L}^n$

$$\mathbf{0x} = \mathbf{0} . \tag{21.2}$$

The function that maps each vector into itself is called the *identity operator* and is denoted by $\mathbf{I}$. Thus for all $\mathbf{x} \in \mathfrak{L}^n$

$$\mathbf{Ix} = \mathbf{x} . \tag{21.3}$$

It is easy to see that $\mathbf{0}$ and $\mathbf{I}$ are linear operators.

We define the operations of addition and multiplication by a scalar, and multiplication in the set of linear operators of $\mathfrak{L}^n$ as follows. If $\mathbf{A}$ and $\mathbf{B}$ are arbitrary linear operators and λ is an arbitrary scalar, the *sum* of **A** and **B** is the function denoted by $\mathbf{A} + \mathbf{B} : \mathfrak{L}^n \to \mathfrak{L}^n$ and defined by

$$(\mathbf{A} + \mathbf{B})\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{x} \tag{21.4}$$

for all $\mathbf{x} \in \mathfrak{L}^n$. The function $\lambda\mathbf{A}$ is defined by

$$(\lambda\mathbf{A})\mathbf{x} = \lambda(\mathbf{A}\mathbf{x}) \tag{21.5}$$

for all $\mathbf{x} \in \mathfrak{L}^n$. The *product* of **A** and **B** is the function denoted by $\mathbf{AB} : \mathfrak{L}^n \to \mathfrak{L}^n$ and defined by

$$(\mathbf{AB})\mathbf{x} = \mathbf{A}(\mathbf{B}\mathbf{x}) \tag{21.6}$$

for all $\mathbf{x} \in \mathfrak{L}^n$ (this is virtually a composite function: the value of **B** is to be taken at **x** and then the value of **A** at $\mathbf{Bx}$). It is left to the reader to show that the sum of two linear operators, the scalar multiple of a linear operator, and the product of two linear operators are again linear operators. It is easy to see that *the linear operators in* $\mathfrak{L}^n$ *form a ring* with respect to addition and multiplication defined among them (see Definition 17.3; this ring is not commutative in general). It can also be seen easily that *the linear operators in* $\mathfrak{L}^n$ *form a linear space* over the real (complex) field with respect to addition and multiplication by a scalar (see Definition 17.4).

DEFINITION 21.2. The linear operator **A** is said to be *non-degenerate* if it maps $\mathfrak{L}^n$ *onto* $\mathfrak{L}^n$, i.e. if its range (the image of its domain $\mathfrak{L}^n$) is $\mathfrak{L}^n$.

THEOREM 21.1. If **A** is a non-degenerate linear operator of $\mathfrak{L}^n$ then it performs a one-to-one mapping of $\mathfrak{L}^n$ onto $\mathfrak{L}^n$.

Proof. Assume, contrary to our proposition, that there are $\mathbf{x}_1, \mathbf{x}_2 \in \mathfrak{L}^n$ such that $\mathbf{x}_1 \neq \mathbf{x}_2$ and $\mathbf{A}\mathbf{x}_1 = \mathbf{A}\mathbf{x}_2$. Introduce the notation $\mathbf{u} = \mathbf{x}_1 - \mathbf{x}_2$. Then obviously $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{Au} = \mathbf{0}$. We can always introduce a base in $\mathfrak{L}^n$ such that one of the base vectors is the given non-zero vector **u**. Since if $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is a base then $\mathbf{u} = \sum_{i=1}^{n} u_i \mathbf{e}_i$ and, **u** being non-zero, at least one of $u_1, \ldots, u_n$ is non-zero. Let, say, $u_n \neq 0$. Then $\mathbf{u}, \mathbf{e}_1, \ldots, \mathbf{e}_{n-1}$ are linearly independent, since if for some $\alpha, \alpha_1, \ldots, \alpha_{n-1}$

$$\alpha\mathbf{u} + \sum_{i=1}^{n-1} \alpha_i \mathbf{e}_i = \mathbf{0},$$

then subtracting

$$\alpha\mathbf{u} - \alpha \sum_{1=i}^{n} u_i \mathbf{e}_i = \mathbf{0}$$

from the previous equality, we get

$$\sum_{i=1}^{n-1} (\alpha_i + \alpha u_i)\mathbf{e}_i + \alpha u_n \mathbf{e}_n = \mathbf{0}.$$

But $\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}, \mathbf{e}_n$ are linearly independent and $u_n \neq 0$. Hence, $\alpha = 0$ and $\alpha_i = 0$ $(i = 1, 2, \ldots, n-1)$ and this proves that $\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}, \mathbf{u}$ form a base.

So let $\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}, \mathbf{u}$ be a base in $\mathfrak{L}^n$. Then for all $\mathbf{x} \in \mathfrak{L}^n$

$$\mathbf{x} = \sum_{k=1}^{n-1} x_k \mathbf{e}_k + \lambda \mathbf{u}$$

and by (21.1) we get

$$\mathbf{A}\mathbf{x} = \mathbf{A}\left(\sum_{k=1}^{n-1} x_k \mathbf{e}_k + \lambda \mathbf{u}\right) = \sum_{k=1}^{n-1} x_k \mathbf{A}\mathbf{e}_k + \lambda \mathbf{A}\mathbf{u}$$
$$= \sum_{k=1}^{n-1} x_k \mathbf{A}\mathbf{e}_k .$$

Thus, the range of **A** is the subspace spanned by the vectors $\mathbf{A}\mathbf{e}_1, \ldots, \mathbf{A}\mathbf{e}_{n-1}$. Since this subspace is of dimension $(n-1)$, at most (see Corollary 9.13 and the remark before Theorem 19.2), the range of **A** cannot be $\mathfrak{L}^n$ and this is a contradiction. !

COROLLARY 21.2. If **A** is a non-degenerate linear operator in $\mathfrak{L}^n$ then there exists one and only one (non-degenerate) linear operator $\mathbf{A}^{-1}$ called the *inverse* of **A**, such that

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I} \tag{21.7}$$

and

$$(\mathbf{A}^{-1})^{-1} = \mathbf{A}. \tag{21.8}$$

Proof. It follows from the previous theorem immediately that the uniquely determined inverse function of **A** exists, its domain is $\mathfrak{L}^n$, and its range is also $\mathfrak{L}^n$. Hence, if this inverse function is denoted by $\mathbf{A}^{-1}$ it follows that (21.7) and (21.8) hold. What is left to prove is that $\mathbf{A}^{-1}$ is a *linear* operator. For arbitrary $\mathbf{y}_1, \mathbf{y}_2 \in \mathfrak{L}^n$ and scalars λ_1, λ_2

$$\mathbf{A}^{-1}\mathbf{y}_1 = \mathbf{x}_1,\ \mathbf{A}^{-1}\mathbf{y}_2 = \mathbf{x}_2,\ \mathbf{A}^{-1}(\lambda_1\mathbf{y}_1 + \lambda_2\mathbf{y}_2) = \mathbf{x}_3$$

if

$$\mathbf{A}\mathbf{x}_1 = \mathbf{y}_1,\ \mathbf{A}\mathbf{x}_2 = \mathbf{y}_2,\ \mathbf{A}\mathbf{x}_3 = \lambda_1\mathbf{y}_1 + \lambda_2\mathbf{y}_2,$$

respectively. But $\lambda_1\mathbf{y}_1 + \lambda_2\mathbf{y}_2 = \mathbf{A}(\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2)$ and since **A** is a one-to-one mapping $\mathbf{x}_3 = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2$. !

We turn now to the problem of how a linear operator can be represented if a base has been introduced in $\mathfrak{L}^n$. If **A** is a linear operator and $\mathbf{e}_1, \ldots, \mathbf{e}_n$ a base in $\mathfrak{L}^n$ then denoting the images of the base vectors in the mapping **A** by

$$\mathbf{a}_k = \mathbf{A}\mathbf{e}_k \qquad (k = 1, 2, \ldots, n) \tag{21.9}$$

the vectors $\mathbf{a}_k$ can be given in the base by

$$\mathbf{a}_k = \sum_{i=1}^{n} a_{ik}\mathbf{e}_i \qquad (k = 1, 2, \ldots, n) \tag{21.10}$$

i.e. $\mathbf{a}_k \simeq a_k$, where a_k is the column vector $a_k = \text{col}\,[a_{1k}, a_{2k}, \ldots, a_{nk}]$. The quadratic matrix $A = [a_{ik}]$ of order n composed from the column vectors $a_1, \ldots, a_n$ is called the *coordinate matrix* of the operator $\mathbf{A}$ in the given base. The fact that A is the coordinate matrix of $\mathbf{A}$ is denoted by writing $\mathbf{A} \simeq A$.

If the coordinate matrix A of the linear operator $\mathbf{A}$ is known, then the column vector representing the image of an arbitrary vector can easily be given provided that the column vector of the latter is known. If $\mathbf{x} \in \mathfrak{L}^n$, $\mathbf{y} = \mathbf{A}\mathbf{x}$ and in the given base $\mathbf{x} \simeq x = \text{col}\,[x_1, \ldots, x_n]$, $\mathbf{y} \simeq y = \text{col}\,[y_1, \ldots, y_n]$, and $\mathbf{A} \simeq A$, then applying (21.1), taking into account (21.9) and (21.10), and interchanging the order of summation, we get

$$\mathbf{y} = \mathbf{A}\mathbf{x} = \mathbf{A}\sum_{k=1}^{n} x_k\mathbf{e}_k = \sum_{k=1}^{n} x_k\mathbf{a}_k = \sum_{k=1}^{n} x_k \sum_{i=1}^{n} a_{ik}\mathbf{e}_i$$

$$= \sum_{i=1}^{n}\left(\sum_{k=1}^{n} a_{ik}x_k\right)\mathbf{e}_i \,.$$

Comparing the last expression with $\mathbf{y} = \sum_{i=1}^{n} y_i\mathbf{e}_i$ we find the result $y_i = \sum_{k=1}^{n} a_{ik}x_k$ $(i = 1, 2, \ldots, n)$, or in matrix notation

$$y = Ax \,. \tag{21.11}$$

If the base is fixed in $\mathfrak{L}^n$ and a quadratic matrix A of order n is arbitrarily given, then (21.11) represents a mapping of $\mathfrak{L}^n$ into $\mathfrak{L}^n$ turning each $\mathbf{x} \in \mathfrak{L}^n$ into the vector $\mathbf{y} \in \mathfrak{L}^n$, whose coordinate column vector is obtained from the coordinate column vector of $\mathbf{x}$ in the given base by (21.11). This mapping is, clearly, a linear operator.

Thus *in a fixed base* the correspondence given by $\mathbf{A} \simeq A$ maps the set of linear operators onto the set of quadratic matrices of order n, and this mapping is one-to-one. An immediate consequence of (21.10) is that in an arbitrary base the image of the zero operator is the zero matrix and that of the identity operator is the unit matrix:

$$\mathbf{0} \simeq O,\ \mathbf{I} \simeq I \,. \tag{21.12}$$

It is also easy to show and is left to the reader that the operations defined among linear operators are represented by operations among their coordinate matrices following the rules of matrix algebra. That is to say, if $\mathbf{A} \simeq A$, $\mathbf{B} \simeq B$ and λ is a scalar

$$\mathbf{A} + \mathbf{B} \simeq A + B,\ \lambda\mathbf{A} \simeq \lambda A,\ \mathbf{AB} \simeq AB \,. \tag{21.13}$$

All results in this Section are, naturally, valid in an n-dimensional Euclidean space. Special linear operators of Euclidean spaces are defined in the two examples below.

EXAMPLE 21.1. *(The outer product operator).* Let $\mathcal{E}^n(\mathcal{R})$ be an n-dimensional real Euclidean space, $\mathbf{a}, \mathbf{b} \in \mathcal{E}^n(\mathcal{R})$, $\mathbf{a} \neq \mathbf{0}$, $b \neq \mathbf{0}$. Then the operator denoted by $\mathbf{a} \circ \mathbf{b}$ is called the *outer product operator* of $\mathbf{a}$ and $\mathbf{b}$ if for all $\mathbf{x} \in \mathcal{E}^n(\mathcal{R})$

$$(\mathbf{a} \circ \mathbf{b})\mathbf{x} = \mathbf{a}(\mathbf{b} \cdot \mathbf{x}) . \tag{21.14}$$

The function $\mathbf{a} \circ \mathbf{b}$ defined by (21.14) is a linear operator, since for arbitrary $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{E}^n(\mathcal{R})$ and $\lambda_1, \lambda_2 \in \mathcal{R}$ we have, applying Definition 18.1, that

$$\begin{aligned}(\mathbf{a} \circ \mathbf{b})(\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2) &= \mathbf{a}[\mathbf{b} \cdot (\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2)] \\ &= \lambda_1\mathbf{a}(\mathbf{b} \cdot \mathbf{x}_1) + \lambda_2\mathbf{a}(\mathbf{b} \cdot \mathbf{x}_2) \\ &= \lambda_1(\mathbf{a} \circ \mathbf{b})\mathbf{x}_1 + \lambda_2(\mathbf{a} \circ \mathbf{b})\mathbf{x}_2 .\end{aligned}$$

The defining formula (21.14) shows that $\mathbf{a} \circ \mathbf{b}$ maps every vector into a vector parallel to $\mathbf{a}$, i.e. it maps the space into a one-dimensional subspace (spanned by $\mathbf{a}$). Thus, if $n \geq 2$, then $\mathbf{a} \circ \mathbf{b}$ is a degenerate operator.

If in the orthonormal base $\mathbf{e}_1, \ldots, \mathbf{e}_n$

$$\mathbf{a} = \sum_{i=1}^{n} a_i\mathbf{e}_i, \qquad \mathbf{b} = \sum_{j=1}^{n} b_j\mathbf{e}_j$$

then

$$(\mathbf{a} \circ \mathbf{b})\mathbf{e}_k = \mathbf{a}(\mathbf{b} \cdot \mathbf{e}_k) = b_k\,\mathbf{a} = \sum_{i=1}^{n} a_i\, b_k\, \mathbf{e}_i, \ (k = 1, 2, \ldots, n).$$

Thus, according to (21.10), the coordinate matrix of $\mathbf{a} \circ \mathbf{b}$ is in the given orthonormal base

$$\mathbf{a} \circ \mathbf{b} \simeq \begin{bmatrix} a_1b_1 & a_1b_2 & \ldots & a_1b_n \\ a_2b_1 & a_2b_2 & \ldots & a_2b_n \\ \cdot & \cdot & \ldots & \cdot \\ a_nb_1 & a_nb_2 & \ldots & a_nb_n \end{bmatrix} \tag{21.15}$$

(cf. (8.15)).

EXAMPLE 21.2. *(The operator* **a** *cross).* Consider the vector $\mathbf{a} \neq \mathbf{0}$ in the ordinary (three-dimensional Euclidean) space. The operator denoted by $(\mathbf{a}\times)$ is called the **a** *cross* operator if for all vectors $\mathbf{x}$ of the ordinary space

$$(\mathbf{a}\times)\mathbf{x} = \mathbf{a}\times\mathbf{x} \tag{21.16}$$

(the cross product of $\mathbf{a}$ and $\mathbf{x}$). It follows from (2.17) and (2.11) that $(\mathbf{a}\times)$ is a linear operator. It turns every vector into a vector orthogonal to $\mathbf{a}$, and thus maps the whole space into the two-dimensional subspace of the vectors orthogonal to **a** (see Lemma 19.4). As a consequence the operator **a** cross is degenerate.

If in the orthonormal (dextral) base $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$

$$\mathbf{a} = \sum_{i=1}^{3} a_i \mathbf{e}_i$$

then

$$\mathbf{a} \times \mathbf{e}_k = \sum_{i=1}^{3} a_i (\mathbf{e}_i \times \mathbf{e}_k) \quad (k = 1, 2, 3) .$$

Hence by (3.4), (3.5), and (3.6)

$$\begin{aligned} \mathbf{a} \times \mathbf{e}_1 &= -a_2 \mathbf{e}_3 + a_3 \mathbf{e}_2 \\ \mathbf{a} \times \mathbf{e}_2 &= a_1 \mathbf{e}_3 - a_3 \mathbf{e}_1 \\ \mathbf{a} \times \mathbf{e}_3 &= -a_1 \mathbf{e}_2 + a_2 \mathbf{e}_1 \end{aligned}$$

and as a consequence in the given base

$$(\mathbf{a} \times) \simeq \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} . \tag{21.17}$$

The degeneracy of a linear operator is equally well represented by its coordinate matrix. We have

THEOREM 21.3. Consider a linear operator **A** in $\mathfrak{L}^n$ and assume that in some base $\mathbf{A} \simeq A$; **A** is non-degenerate if and only if the matrix A is regular.

Proof. This theorem is an immediate consequence of Theorem **21.1** if we take into account that for an arbitrary column vector y the linear system (21.11) has one and only one solution x if and only if A is regular (see Theorem **13.3** and Corollary **14.2**). !

From what we have said about the representation of linear operators with matrices, it follows that if **A** is a non-degenerate linear operator in $\mathfrak{L}^n$ and $\mathbf{A} \simeq A$ in some base, then in the same base

$$\mathbf{A}^{-1} \simeq A^{-1} . \tag{21.18}$$

An important problem to be settled is how the coordinate matrix of a linear operator is transformed if a transformation of the base takes place. Consider a linear operator **A** in $\mathfrak{L}^n$ and two arbitrary bases, the old one $\mathbf{e}_1, \ldots, \mathbf{e}_n$ and the new one $\tilde{\mathbf{e}}_1, \ldots, \tilde{\mathbf{e}}_n$. Assume that the matrix of the base transformation is $T = [t_{ik}]$ (see (20.1)). Let $\mathbf{A} \simeq A$ and $\mathbf{A} \simeq \tilde{A}$ in the old and in the new base, respectively. For $\mathbf{x} \in \mathfrak{L}^n$, $\mathbf{y} = \mathbf{A}\mathbf{x}$ let $\mathbf{x} \simeq x$, $\mathbf{y} \simeq y$ and $\mathbf{x} \simeq \tilde{x}$, $\mathbf{y} \simeq \tilde{y}$ in the old and in the new base, respectively. By (21.11) we have

$$y = Ax, \quad \tilde{y} = \tilde{A}\tilde{x} . \tag{21.19}$$

Substituting (20.3) and the corresponding formula for y into the first equation of (21.19) we get

$$T\tilde{y} = AT\tilde{x} ,$$

or, multiplying this by T^{-1} from the left,

$$\tilde{y} = (T^{-1}AT)\tilde{x} .$$

Subtracting the last equation from the second one of (21.19) we see that for all $\tilde{x}$

$$\widetilde{A}\tilde{x} - (T^{-1}AT)\tilde{x} = (\widetilde{A} - T^{-1}AT)\tilde{x} = 0$$

and this is true if and only if $\widetilde{A} - T^{-1}AT$ is the zero matrix, i.e. if

$$\widetilde{A} = T^{-1}AT . \tag{21.20}$$

Thus, the transformation formula for the coordinate matrix of a linear operator **A** is (21.20) where T is the matrix of the base transformation. In other words the relation between two matrices representing the same linear operator in different bases is(21.20).

Definition 21.3. We say that the quadratic matrices A and B of order n are *similar*, $A \simeq B$, if there is a regular matrix T of order n such that

$$B = T^{-1}AT . \tag{21.21}$$

Note that the similarity of nth-order matrices is an *equivalence relation*. This means that it has the following properties. It is *reflexive*, i.e. $A \simeq A$, since with the unit matrix $I^{-1}AI = A$. It is *symmetric*, i.e. if $A \simeq B$ then $B \simeq A$, since multiplying (21.21) by T from the left and by T^{-1} from the right

$$A = TBT^{-1} = (T^{-1})^{-1}BT^{-1} .$$

Finally, it is *transitive*, i.e. if $A \simeq B$ and $B \simeq C$ then $A \simeq C$, since substituting (21.21) into $C = S^{-1}BS$ (S is an appropriate regular matrix) we get

$$C = S^{-1}T^{-1}ATS = (TS)^{-1}A(TS) .$$

An immediate consequence of Theorem 12.8 is that similar matrices are equivalent (in the sense of Definition 12.2), i.e. their ranks are equal.

By Definition 21.3 the transformation formula (21.20) implies that all the matrices representing a linear operator in different bases are similar.

In the important particular case when the space is the n-dimensional real Euclidean space $\mathscr{E}^n(\mathscr{R})$ and the base *transformation is orthogonal*, i.e. both the old and the new bases are orthonormal, the matrix of the transformation is an orthogonal matrix M (see Definition 20.1) and the transformation formula (21.20) assumes the form

$$\widetilde{A} = M'AM . \tag{21.22}$$

22. Eigenvalues and Eigenvectors

A linear operator maps the space $\mathfrak{L}^n$ into itself; at the same time it may happen that $\mathfrak{L}^n$ has a proper subspace (i.e. a subspace that is neither $\mathfrak{L}^n$ nor the empty set) that is mapped by the operator into itself. The study of such subspaces helps to understand the structure of a linear operator.

Definition 22.1. A subspace $\mathfrak{L}^m$ of the linear space $\mathfrak{L}^n$ is called an *invariant subspace* of the linear operator $\mathbf{A}$ if it is mapped by $\mathbf{A}$ into itself, that is to say, if for all $\mathbf{x} \in \mathfrak{L}^m$ also $\mathbf{Ax} \in \mathfrak{L}^m$.

The space $\mathfrak{L}^n$ and the subspace whose only element is the zero vector are, clearly, invariant subspaces of any linear operator. The most important invariant subspaces are those of one dimension. We approach these through the following:

Definition 22.2. The (real or complex) scalar number λ and the vector $\mathbf{s} \in \mathfrak{L}^n$ are called an *eigenvalue* and a corresponding *eigenvector* of the linear operator $\mathbf{A}$ if

$$\mathbf{As} = \lambda\mathbf{s} \text{ and } \mathbf{s} \neq \mathbf{0}\,, \tag{22.1}$$

i.e. the non-zero vector $\mathbf{s}$ is an eigenvector of $\mathbf{A}$ if its image is a scalar multiple of $\mathbf{s}$, and the scalar λ is an eigenvalue if there is a non-zero vector $\mathbf{s}$ for which (22.1) holds (cf. Definition 15.1).

If a base is introduced in $\mathfrak{L}^n$ and in this base $\mathbf{A} \simeq A$ and $\mathbf{s} \simeq s$ then (22.1) assumes the coordinate form (15.4). This means that λ and $\mathbf{s}$ are an eigenvalue and a corresponding eigenvector of the linear operator $\mathbf{A}$ if and only if λ and the coordinate column vector s of $\mathbf{s}$ are an eigenvalue and a corresponding eigenvector of the coordinate matrix A of $\mathbf{A}$. As a consequence we get from Theorem 15.7 the following:

Theorem 22.1. Let $\mathfrak{L}^n$ be a linear space over the complex field, $\mathbf{A}$ a linear operator, and $\mathbf{A} \simeq A$ in a given base; λ is an eigenvalue of $\mathbf{A}$ if and only if it is a solution of the characteristic equation

$$\det\,(A - \lambda I) = 0. \tag{22.2}$$

If $\mathfrak{L}^n$ is a linear space over the real field (the space $\mathcal{E}^n(\mathfrak{R})$, say) then only real scalars may be eigenvalues of a linear operator and, as a consequence, only the 'only if' part of the previous theorem remains valid. In this case the left-hand side of (22.2) is a polynomial of degree n with real coefficients; but (22.2) may have complex roots and these cannot be considered proper eigenvalues of a linear operator in a real space. However, for convenience, the complex roots of (22.2) will also be called *eigenvalues* of the linear operator and so Theorem 22.1 remains formally valid for linear spaces over the real field.

In different bases a linear operator is represented by different quadratic matrices. If λ is an eigenvalue of one of these coordinate matrices then it is an eigenvalue of the operator and, consequently, of any matrix representing the same operator. In fact, as we know from (21.20) the coordinate matrices representing the same linear operator in different bases are all similar and we have

THEOREM 22.2. Similar matrices have equal eigenvalues with equal multiplicities.

Proof. We show that the characteristic polynomials of similar matrices are equal. This implies our proposition. If $\tilde{A} = T^{-1}AT$, where T is a regular matrix, then applying Theorems 10.16 and **11.3** we have

$$\begin{aligned}\det(\tilde{A} - \lambda I) &= \det(T^{-1}AT - \lambda T^{-1}IT)\\ &= \det(T^{-1}(A - \lambda I)\,T)\\ &= \det T^{-1}\,\det(A - \lambda I)\det T\\ &= \det(A - \lambda I).\,!\end{aligned}$$

COROLLARY 22.3. A linear operator of the space $\mathfrak{L}^n$ has n different eigenvalues, at most.

DEFINITION 22.3. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the nth-order matrix A (an eigenvalue occurs here the same number of times as its multiplicity); the sum of the eigenvalues, i.e. the number

$$\operatorname{Tr} A = \sum_{k=1}^{n} \lambda_k,$$

is called the *trace* of the matrix A.

According to Theorem 7.8 and (5.25) the trace of a quadratic matrix with real elements is always real. We get from Theorem 22.2:

COROLLARY 22.4. The traces of similar matrices are equal.

We also have the following:

THEOREM 22.5. The trace of the matrix $A = [a_{ik}]$ of order n is equal to the sum of the the elements occurring in the main diagonal; that is,

$$\operatorname{Tr} A = \sum_{k=1}^{n} a_{kk}\,.$$

Proof. This proposition follows at once from Vietà's first formula (see (7.8)) if we take into account that

$$\det(A - \lambda I) = (-1)^n\,\lambda^n + (-1)^{n-1}\,\lambda^{n-1}\sum_{k=1}^{n} a_{kk} + \ldots + \det A.\,! \tag{22.3}$$

Returning to the original problem of invariant subspaces we have

THEOREM 22.6. If λ is an eigenvalue and $\mathbf{s}$ a corresponding eigenvector of the linear operator $\mathbf{A}$, then for any non-zero scalar μ the vector $\mu\mathbf{s}$ is also an eigenvector (corresponding to the same eigenvalue λ) and the one-dimensional subspace spanned by $\mathbf{s}$

$$\mathfrak{L}^1 = \{\mathbf{x} \in \mathfrak{L}^n : \mathbf{x} = \mu\mathbf{s},\ \mu \in \mathfrak{R} \text{ or } \mathfrak{C}\}$$

is an invariant subspace of $\mathbf{A}$.

Proof.

$$\mathbf{A}(\mu\mathbf{s}) = \mu\mathbf{A}\mathbf{s} = \mu\lambda\mathbf{s} = \lambda(\mu\mathbf{s}) \in \mathfrak{L}^1 \,.\,!$$

THEOREM 22.7. Let $\mathbf{A}$ be a linear operator in the linear space $\mathfrak{L}^n$ over the real field $\mathfrak{R}$, and $\lambda = \alpha + i\beta$ $(\alpha, \beta \in \mathfrak{R})$ a complex eigenvalue of $\mathbf{A}$; then there are two vectors $\mathbf{u}, \mathbf{v} \in \mathfrak{L}^n$, not both zero, such that

$$\mathbf{A}\mathbf{u} = \alpha\mathbf{u} - \beta\mathbf{v},\ \ \mathbf{A}\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{u}\,. \tag{22.4}$$

Proof. Introduce a base in $\mathfrak{L}^n$ and let $\mathbf{A} \simeq A$ in this base. The linear system

$$(A - \lambda I)\, s = 0$$

with $\lambda = \alpha + i\beta$ has non-trivial solutions, since its determinant is zero. Let $s = u + iv$ be a non-trivial solution where u and v are column vectors with real elements and at least one of $u \neq 0$, $v \neq 0$ holds. Thus

$$(A - (\alpha + i\beta)I)\ (u + iv) = 0;$$

hence

$$\begin{aligned} Au + iAv &= (\alpha + i\beta)\,(u + iv) \\ &= \alpha u - \beta v + i(\alpha v + \beta u)\,. \end{aligned}$$

Equating real and imaginary parts

$$Au = \alpha u - \beta v,\ Av = \alpha v + \beta u\,. \tag{22.5}$$

From the transformation formulae (20.3) and (21.20) it follows that (22.5) holds in an arbitrary base for the coordinate column vectors of the vectors $\mathbf{u}, \mathbf{v}$ that are determined by $\mathbf{u} \simeq u$, $\mathbf{v} \simeq v$ in the given base. This implies (22.4). !

COROLLARY 22.8. If $\lambda = \alpha + i\beta$ $(\alpha, \beta \in \mathfrak{R})$ is a complex eigenvalue of the linear operator $\mathbf{A}$ of the linear space $\mathfrak{L}^n$ over the real field $\mathfrak{R}$, then it determines an invariant subspace of dimension two (or one).

Proof. By the previous theorem two vectors $\mathbf{u}$ and $\mathbf{v}$ can be found such that at least one of them is non-zero and (22.4) holds. We show that the subspace $\mathfrak{L}^2$ spanned by $\mathbf{u}$ and $\mathbf{v}$ (which is, in general, of dimension 2 but may be one-dimensional) is an invariant subspace of $\mathbf{A}$. For if $\mathbf{x} \in \mathfrak{L}^2$, i.e.

$$\mathbf{x} = \mu_1 \mathbf{u} + \mu_2 \mathbf{v}, \ \mu_1, \mu_2 \in \mathfrak{R}$$

then applying (22.4)

$$\begin{aligned}\mathbf{Ax} &= \mathbf{A}(\mu_1 \mathbf{u} + \mu_2 \mathbf{v}) = \mu_1 \mathbf{Au} + \mu_2 \mathbf{Av} \\ &= \mu_1(\alpha\mathbf{u} - \beta\mathbf{v}) + \mu_2(\alpha\mathbf{v} + \beta\mathbf{u}) \\ &= (\mu_1 \alpha + \mu_2 \beta)\, \mathbf{u} + (\mu_2 \alpha - \mu_1 \beta)\, \mathbf{v}\,;\end{aligned}$$

thus $\mathbf{Ax} \in \mathfrak{L}^2$. !

23. Symmetric Operators

In this Section we deal with some important types of operators in the n-dimensional Euclidean space. We concentrate on the real Euclidean space $\mathcal{E}^n(\mathfrak{R})$ and refer only occasionally to similar ideas in the complex Euclidean space.

Definition 23.1. The linear operator $\mathbf{A}^*$ is called the *adjoint* of the linear operator $\mathbf{A}$ in $\mathcal{E}^n(\mathfrak{R})$ if, for all $\mathbf{x}, \mathbf{y} \in \mathcal{E}^n(\mathfrak{R})$,

$$(\mathbf{Ax}) \cdot \mathbf{y} = \mathbf{x} \cdot (\mathbf{A}^* \mathbf{y}). \tag{23.1}$$

The definition of the adjoint operator is the same in the complex Euclidean space.

Note that if $\mathbf{A}^*$ is the adjoint of $\mathbf{A}$ then, obviously, $\mathbf{A}$ is the adjoint of $\mathbf{A}^*$, i.e. $(\mathbf{A}^*)^* = \mathbf{A}$. It is also easy to see from the definition that for the adjoint of the product of operators we have

$$(\mathbf{AB})^* = \mathbf{B}^* \mathbf{A}^* . \tag{23.2}$$

Theorem 23.1. Each linear operator in $\mathcal{E}^n(\mathfrak{R})$ has one and only one adjoint.

Proof. The uniqueness of the adjoint will be proved first. Let $\mathbf{A}$ be an arbitrary linear operator in $\mathcal{E}^n(\mathfrak{R})$ and assume that for some $\mathbf{A}^*$ the identity (23.1) is valid. If $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is an orthonormal base in $\mathcal{E}^n(\mathfrak{R})$, then according to (19.10) for each $\mathbf{y} \in \mathcal{E}^n(\mathfrak{R})$

$$\mathbf{A}^* \mathbf{y} = \sum_{k=1}^{n} [(\mathbf{A}^* \mathbf{y}) \cdot \mathbf{e}_k]\, \mathbf{e}_k \, .$$

Applying (23.1) and property a) of Definition 18.1 we get

$$\mathbf{A}^* \mathbf{y} = \sum_{k=1}^{n} [\mathbf{y} \cdot (\mathbf{Ae}_k)]\, \mathbf{e}_k \, . \tag{23.3}$$

For a given $\mathbf{A}$, the right-hand side of (23.3) determines the image of each vector $\mathbf{y}$ uniquely; thus, if the adjoint $\mathbf{A}^*$ exists, it is uniquely determined. Conversely, if $\mathbf{A}^*$ is defined by (23.3) then, by the basic properties of the scalar product, it is a linear operator and for all

$$\mathbf{x} = \sum_{k=1}^{n} x_k \mathbf{e}_k, \qquad \mathbf{y} = \sum_{i=1}^{n} y_i \mathbf{e}_i$$

we have, applying (19.11),

$$\begin{aligned}\mathbf{x} \cdot (\mathbf{A}^* \mathbf{y}) &= \sum_{k=1}^{n} x_k [\mathbf{y} \cdot (\mathbf{A}\mathbf{e}_k)] = \sum_{k=1}^{n} x_k \left[\left(\sum_{i=1}^{n} y_i \mathbf{e}_i\right) \cdot (\mathbf{A}\mathbf{e}_k)\right] \\ &= \sum_{i,k=1}^{n} x_k y_i [\mathbf{e}_i \cdot (\mathbf{A}\mathbf{e}_k)] = \sum_{i=1}^{n} y_i \left(\mathbf{e}_i \cdot \sum_{k=1}^{n} x_k \mathbf{A}\mathbf{e}_k\right) \\ &= \sum_{i=1}^{n} y_i \mathbf{e}_i \cdot (\mathbf{A}\mathbf{x}) = \left(\sum_{i=1}^{n} y_i \mathbf{e}_i\right) \cdot (\mathbf{A}\mathbf{x}) = (\mathbf{A}\mathbf{x}) \cdot \mathbf{y}\,. \,!\end{aligned}$$

A similar theorem can also be proved in $\mathcal{E}^n(\mathcal{C})$.

THEOREM 23.2. If $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is an orthonormal base in $\mathfrak{L}^n(\mathfrak{R})$ and in this base $\mathbf{A} \simeq A$, then for the adjoint operator we have $\mathbf{A}^* \simeq A'$ where A' is the transpose of the coordinate matrix A.

Proof. If in the given orthonormal base $\mathbf{A}^* \simeq B$ and $\mathbf{x} \simeq x$, $\mathbf{y} \simeq y$ then writing (23.1) in coordinates

$$(Ax)'\,y = x'\,By$$

or

$$x'\,A'\,y = x'\,By$$

and hence

$$x'(A' - B)\,y = 0.$$

For a fixed y this means that $(A' - B)y$ is orthogonal to each vector x, and thus $(A' - B)y = 0$. Since the last identity holds for each column vector y, it follows that $A' - B$ is the zero matrix, i.e. $B = A'$. !

An analogous theorem holds in $\mathcal{E}^n(\mathcal{C})$ with the adjoint matrix A^* playing the role of the transpose (see Definition 8.4).

DEFINITION 23.2. The linear operator $\mathbf{S}$ of $\mathcal{E}^n(\mathfrak{R})$ is said to be *symmetric* (or *self-adjoint*) if $\mathbf{S}^* = \mathbf{S}$ or, in other words, if for all $\mathbf{x}, \mathbf{y} \in \mathcal{E}^n(\mathfrak{R})$

$$(\mathbf{S}\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (\mathbf{S}\mathbf{y})\,. \tag{23.4}$$

THEOREM 23.3. A linear operator $\mathbf{S}$ of $\mathcal{E}^n(\mathfrak{R})$ is symmetric if and only if its coordinate matrix is symmetric in an arbitrary orthonormal base (see Definition 8.6).

Proof. It follows at once from Theorem 23.2 that if $\mathbf{S}$ is symmetric and in an orthonormal base $\mathbf{S} \simeq S$ then $S' = S$. Assume, conversely, that in some orthonormal base $\mathbf{S} \simeq S$ and $S' = S$. We show first that the coordinate matrix of **S** is symmetric in every orthonormal base. If in some other orthonormal base $\mathbf{S} \simeq \tilde{S}$ and M is the orthogonal matrix of the base transformation, then by (21.22)

$$\tilde{S} = M' S M \ .$$

Transposing both sides of this equality we get

$$\tilde{S}' = (M' S M)' = M' S' (M')' = M' S M = \tilde{S} \ .$$

Now, the identity (23.4) is valid, for in an arbitrary orthonormal base applying (19.11)

$$(\mathbf{S}\mathbf{x}) \cdot \mathbf{y} = (Sx)' y = x' S' y = x' S y = \mathbf{x} \cdot (\mathbf{S}\mathbf{y}). \, !$$

In $\mathcal{E}^n(\mathcal{C})$ self-adjoint operators are also called hermitian, and a theorem analogous to the previous one holds with hermitian matrices playing the role of the symmetric matrices.

DEFINITION 23.3. The linear operator **Z** of $\mathcal{E}^n(\mathcal{R})$ is said to be *skew-symmetric* if $\mathbf{Z}^* = -\mathbf{Z}$ or, in other words, if for all $\mathbf{x}, \mathbf{y} \in \mathcal{E}^n(\mathcal{R})$

$$(\mathbf{Z}\mathbf{x}) \cdot \mathbf{y} = - \mathbf{x} \cdot (\mathbf{Z}\mathbf{y}) \ . \tag{23.5}$$

The proof of the following theorem is similar to the proof of Theorem 23.3 and is therefore left to the reader.

THEOREM 23.4. A linear operator Z of $\mathcal{E}^n(\mathcal{R})$ is skew-symmetric if and only if its coordinate matrix is skew-symmetric in every orthonormal base (see Definition 8.7).

EXAMPLE 23.1. In Example 21.1 the outer product operator $\mathbf{a} \circ \mathbf{b}$ has already been discussed. If $\mathbf{a}, \mathbf{b} \in \mathcal{E}^n(\mathcal{R})$, $\mathbf{a} \neq \mathbf{0}$, $\mathbf{b} \neq \mathbf{0}$ then for any $\mathbf{x} \in \mathcal{E}^n(\mathcal{R})$, $(\mathbf{a} \circ \mathbf{b})\, \mathbf{x} = \mathbf{a}(\mathbf{b} \cdot \mathbf{x})$. Applying the concepts introduced in the previous Section, we see that **a** is an eigenvector of this operator with the eigenvalue $\mathbf{a} \cdot \mathbf{b}$ since

$$(\mathbf{a} \circ \mathbf{b})\, \mathbf{a} = \mathbf{a}(\mathbf{b} \cdot \mathbf{a}) \ .$$

All vectors orthogonal to **b** are also eigenvectors with the eigenvalue 0 since if **x** is orthogonal to **b** then

$$(\mathbf{a} \circ \mathbf{b})\, \mathbf{x} = \mathbf{a}\ (\mathbf{b} \cdot \mathbf{x}) = 0 = 0\mathbf{x}.$$

Thus if **a** is not orthogonal to **b** then $\mathbf{a} \circ \mathbf{b}$ has two eigenvalues $\mathbf{a} \cdot \mathbf{b}$ and zero. The $(n-1)$-dimensional subspace of the vectors that are orthogonal to **b** is an invariant subspace (see Lemma 19.4). Since all vectors of this subspace are eigenvectors and $n-1$ non-zero vectors can be found in this subspace, any two of which are orthogonal, $\mathbf{a} \circ \mathbf{b}$ has $n-1$ eigenvectors that are orthogonal to each other. In the particular case when **a** is parallel to **b**, i.e. $\mathbf{b} = \alpha\mathbf{a}$ for some $\alpha \in \mathcal{R}$, **a** is an eigenvector which is orthogonal to all these eigenvectors. Thus in this particular case $\mathbf{a} \circ (\alpha\mathbf{a})$ has an orthogonal set of n eigenvectors.

In general, for the adjoint $(\mathbf{a} \circ \mathbf{b})^*$ of $\mathbf{a} \circ \mathbf{b}$ we have the identity (23.1):

$$\mathbf{x} \cdot (\mathbf{a} \circ \mathbf{b})^* \mathbf{y} = (\mathbf{a} \circ \mathbf{b})\, \mathbf{x} \cdot \mathbf{y} = \big(\mathbf{a}(\mathbf{b} \cdot \mathbf{x})\big) \cdot \mathbf{y} = (\mathbf{x} \cdot \mathbf{b})\,(\mathbf{a} \cdot \mathbf{y})$$

$$= \mathbf{x} \cdot (\mathbf{b} \circ \mathbf{a})\mathbf{y}, \qquad \mathbf{x}, \mathbf{y} \in \mathcal{E}^n(\mathcal{R})\,.$$

Thus $(\mathbf{a} \circ \mathbf{b})^* = \mathbf{b} \circ \mathbf{a}$. In the particular case when $\mathbf{b} = \alpha\mathbf{a}$ we have $\big(\mathbf{a} \circ (\alpha\mathbf{a})\big)^* = (\alpha\mathbf{a}) \circ \mathbf{a} = \mathbf{a} \circ (\alpha\mathbf{a})$ and this means that $\mathbf{a} \circ (\alpha\mathbf{a})$ is symmetric. In fact, its coordinate matrix (21.15) is now

$$\mathbf{a} \circ (\alpha\mathbf{a}) \simeq \alpha \begin{bmatrix} a_1^2 & a_1 a_2 & \dots & a_1 a_n \\ a_2 a_1 & a_2^2 & \dots & a_2 a_n \\ \cdot & \cdot & \dots & \cdot \\ a_n a_1 & a_n a_2 & \dots & a_n^2 \end{bmatrix}$$

and this is a symmetric matrix.

EXAMPLE 23.2. In Example 21.2 the $(\mathbf{a}\times)$ operator has already been discussed ($\mathbf{a}$ is a non-zero vector in the ordinary space). The two-dimensional subspace of the vectors that are orthogonal to $\mathbf{a}$ is an invariant subspace, since $(\mathbf{a}\times)$ maps the whole space including this subspace onto this subspace (for any $\mathbf{x}$ the image $\mathbf{a}\times\mathbf{x}$ is a vector orthogonal to $\mathbf{a}$). The vector $\mathbf{a}$ is an eigenvector with eigenvalue 0 since $\mathbf{a}\times\mathbf{a} = 0 = 0\mathbf{a}$.

For the adjoint $(\mathbf{a}\times)^*$ we have the identity (23.1):

$$\mathbf{x} \cdot (\mathbf{a}\times)^*\mathbf{y} = (\mathbf{a}\times\mathbf{x}) \cdot \mathbf{y} = (\mathbf{axy}) = -(\mathbf{xay}) = \mathbf{x} \cdot (-\mathbf{a}\times\mathbf{y})$$

where property (2.16) of the scalar triple product has been applied. Hence $(\mathbf{a}\times)^* = -(\mathbf{a}\times)$ and $(\mathbf{a}\times)$ is a skew-symmetric linear operator (cf. also (21.17) and Theorem 23.4).

In the rest of this Section we consider eigenvalues and eigenvectors of symmetric operators.

THEOREM 23.5. An arbitrary symmetric operator of $\mathcal{E}^n(\mathcal{R})$ has exactly n real eigenvalues if each eigenvalue is counted with its multiplicity; in other words, a symmetric operator has no complex eigenvalues.

Proof. Assume that $\lambda = \alpha + i\beta$ is a complex eigenvalue of the symmetric operator $\mathbf{S}$. Then, by Theorem 22.7, two vectors $\mathbf{u}, \mathbf{v} \in \mathcal{E}^n(\mathcal{R})$ can be found such that $\|\mathbf{u}\| + \|\mathbf{v}\| > 0$ and

$$\mathbf{Su} = \alpha\mathbf{u} - \beta\mathbf{v}, \ \mathbf{Sv} = \alpha\mathbf{v} + \beta\mathbf{u}.$$

Take the scalar product of both sides of the first equality with $\mathbf{v}$ and the scalar product of both sides of the second equality with $\mathbf{u}$:

$$\mathbf{v} \cdot (\mathbf{Su}) = \alpha\mathbf{u} \cdot \mathbf{v} - \beta\|\mathbf{v}\|^2$$

$$\mathbf{u} \cdot (\mathbf{Sv}) = \alpha\mathbf{u} \cdot \mathbf{v} + \beta\|\mathbf{u}\|^2\,.$$

Subtracting the first equality from the second we obtain by (23.4)

$$\beta(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2) = 0 .$$

Hence $\beta = 0$ and λ is real. !

COROLLARY 23.6. A symmetric operator in $\mathscr{E}^n(\mathscr{R})$ has at least one (real) eigenvector.

THEOREM 23.7. If $\mathbf{S}$ is a symmetric operator in $\mathscr{E}^n(\mathscr{R})$, the real numbers λ_1 and λ_2 are eigenvalues with respective eigenvectors $\mathbf{s}_1$ and $\mathbf{s}_2$ and $\lambda_1 \neq \lambda_2$ then $\mathbf{s}_1$ and $\mathbf{s}_2$ are orthogonal (i.e. eigenvectors corresponding to different eigenvalues of a symmetric operator are orthogonal).

Proof. According to the assumptions

$$\mathbf{S}\mathbf{s}_1 = \lambda_1 \mathbf{s}_1, \ \mathbf{S}\mathbf{s}_2 = \lambda_2 \mathbf{s}_2 .$$

Taking the scalar product of both sides of the first equality with $\mathbf{s}_2$ and of the second with $\mathbf{s}_1$ we get

$$\mathbf{s}_2 \cdot (\mathbf{S}\mathbf{s}_1) = \lambda_1 \mathbf{s}_1 \cdot \mathbf{s}_2, \ \ \mathbf{s}_1 \cdot (\mathbf{S}\mathbf{s}_2) = \lambda_2 \mathbf{s}_1 \cdot \mathbf{s}_2 .$$

Subtracting the first equality from the second and applying (23.4) we have

$$(\lambda_2 - \lambda_1)\mathbf{s}_1 \cdot \mathbf{s}_2 = 0 .$$

Since $\lambda_1 \neq \lambda_2$ the condition $\mathbf{s}_1 \cdot \mathbf{s}_2 = 0$ must hold. !

COROLLARY 23.8. If a symmetric operator of $\mathscr{E}^n(\mathscr{R})$ has n different eigenvalues (all roots of the characteristic polynomial are simple) then it has an orthogonal system of n eigenvectors.

We now show that the proposition expressed in this corollary is true not only in the particular case of different eigenvalues but also in the general case.

LEMMA 23.9. If $\mathbf{s}$ is an eigenvector of the symmetric operator $\mathbf{S}$ and

$$\mathscr{E}^{n-1} = \{\mathbf{x} \in \mathscr{E}^n(\mathscr{R}) : \mathbf{x} \cdot \mathbf{s} = 0\}$$

is the $(n-1)$-dimensional subspace of the vectors orthogonal to $\mathbf{s}$ (see Lemma 19.4), then $\mathscr{E}^{n-1}$ is an invariant subspace of $\mathbf{S}$.

Proof. If $\mathbf{S}\mathbf{s} = \lambda\mathbf{s}$ and $\mathbf{x} \in \mathscr{E}^{n-1}$, i.e. $\mathbf{x} \cdot \mathbf{s} = 0$ then by (23.4)

$$(\mathbf{S}\mathbf{x}) \cdot \mathbf{s} = \mathbf{x} \cdot (\mathbf{S}\mathbf{s}) = \mathbf{x} \cdot (\lambda\mathbf{s}) = \lambda\mathbf{x} \cdot \mathbf{s} = 0,$$

thus $\mathbf{S}\mathbf{x} \in \mathscr{E}^{n-1}$. !

Now we can prove the basic

THEOREM 23.10. *(Main Diagonal Theorem)*. If $\mathbf{S}$ is a symmetric operator in $\mathscr{E}^n(\mathscr{R})$ then it has an orthogonal system of n unit eigenvectors and in the orthonormal base of these vectors the coordinate matrix of **S** is diagonal.

Proof. According to Corollary 23.6, **S** has an eigenvector. Thus, applying also Theorem 22.6, a unit eigenvector $\mathbf{e}_1$ can be chosen ($\|\mathbf{e}_1\| = 1$). According to Lemma 19.4 the vectors orthogonal to $\mathbf{e}_1$ form an $(n-1)$-dimensional subspace $\mathscr{E}^{n-1}$ of $\mathscr{E}^n(\mathfrak{R})$ and this is, by Lemma 23.9, an invariant subspace of **S**. Obviously the restriction of **S** to $\mathscr{E}^{n-1}$ (i.e. **S** considered as an operator in the subspace $\mathscr{E}^{n-1}$) is a symmetric operator of $\mathscr{E}^{n-1}$ and so has a unit eigenvector: $\mathbf{e}_2 \in \mathscr{E}^{n-1}$, $\|\mathbf{e}_2\| = 1$, and $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$. Consider the subspace $\mathscr{E}^{n-2}$ of $\mathscr{E}^{n-1}$ formed by the vectors in $\mathscr{E}^{n-1}$ that are orthogonal to $\mathbf{e}_2$. This is an $(n-2)$-dimensional invariant subspace of **S**. The operator **S** is symmetric in $\mathscr{E}^{n-2}$; thus, it has there a unit eigenvector: $\mathbf{e}_3 \in \mathscr{E}^{n-2}$, $\|\mathbf{e}_3\| = 1$ and $\mathbf{e}_2 \cdot \mathbf{e}_3 = \mathbf{e}_1 \cdot \mathbf{e}_3 = 0$. Proceeding like this, an orthonormal set of n unit eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ can be constructed in n steps. That is,

$$\mathbf{e}_i \cdot \mathbf{e}_k = \delta_{ik}, \quad \mathbf{S}\mathbf{e}_k = \lambda_k \mathbf{e}_k, \qquad (i, k = 1, 2, \ldots, n)$$

where the real numbers λ_k are not necessarily different eigenvalues of **S**. This set of eigenvectors can be chosen for an orthonormal base in $\mathscr{E}^n(\mathfrak{R})$. Formulae (21.9) and (21.10) defining the coordinate matrix of **S** in the given base are now

$$\mathbf{a}_k = \mathbf{S}\mathbf{e}_k = \lambda_k \mathbf{e}_k, \qquad (k = 1, 2, \ldots, n)\,;$$

thus for the coordinate matrix S we have

$$\mathbf{S} \simeq S = \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ 0 & 0 & \lambda_3 & \ldots & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix} .\,! \qquad (23.6)$$

Note that the orthonormal base of eigenvectors constructed in the proof of the Main Diagonal Theorem is not uniquely determined. The permutation of the base vectors is always possible ($n \geq 2$) and, in general, several different orthonormal bases of eigenvectors can be chosen. However, as we shall now show, the elements in the main diagonal of the matrix (23.6) are the eigenvalues, each occurring the same number of times as its multiplicity. In fact, we have the following Corollary which is in a certain sense the converse of the Main Diagonal Theorem:

COROLLARY 23.11. If **A** is a linear operator, $\mathbf{e}_1, \ldots, \mathbf{e}_n$ an orthonormal base in $\mathscr{E}^n(\mathfrak{R})$, and the coordinate matrix of **A** in this base is the diagonal matrix

$$A = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

($\lambda_k \in \mathfrak{R}$) then **A** is a symmetric operator, the vectors $\mathbf{e}_k$ are its eigenvectors, and λ_k ($k = 1, 2, \dots, n$) are its eigenvalues, the same λ_k occurring the number of times corresponding to its multiplicity.

Proof. Since a diagonal matrix is symmetric the first proposition follows from Theorem 23.3. Applying (21.9) and (21.10) with our diagonal coordinate matrix A we get

$$\mathbf{A}\mathbf{e}_k = \lambda_k \mathbf{e}_k \quad (k = 1, 2, \dots, n),$$

i.e. $\mathbf{e}_k$ and λ_k are eigenvectors and eigenvalues of **A**, respectively. The last statement follows from the fact that the characteristic equation of the operator **A** (see Theorem 22.1) is now

$$\det (A - \lambda I) = \begin{bmatrix} \lambda_1 - \lambda & 0 & \dots & 0 \\ 0 & \lambda_2 - \lambda & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & \lambda_n - \lambda \end{bmatrix}$$

$$= \prod_{k=1}^{n} (\lambda_k - \lambda) = 0 \,. \,!$$

The Main Diagonal Theorem can be restated for symmetric matrices as follows:

Corollary 23.12. If A is a real symmetric matrix then an orthogonal matrix M can be found such that

$$D = M' AM = M^{-1} AM \tag{23.7}$$

is diagonal (A is similar to a diagonal matrix); the column vectors of M are eigenvectors of A and the elements in the main diagonal of D are the eigenvalues of A (see Definition 20.1 and Theorem 22.2).

Proof. The symmetric matrix A can be considered as the coordinate matrix of some symmetric operator **A** in an orthonormal base. By the Main Diagonal Theorem an orthonormal base exists such that in it the coordinate matrix of **A** is a diagonal matrix D. Formula (21.22) for the orthogonal base transformation yields (23.7). Multiply the equality (23.7) by M from the left:

$$MD = AM \,.$$

If $M = [m_1, \dots, m_n]$ and $D = [\delta_{ik} \lambda_i]$ then it is easy to see that the kth column vector of the matrix on the left-hand side of the last equality is $\lambda_k m_k$:

$$Am_k = \lambda_k m_k \quad (k = 1, 2, \dots, n) \,. \,! \tag{23.8}$$

If **S** is a symmetric operator in $\mathcal{E}^n(\mathcal{R})$,

$$\mathbf{S}\mathbf{e}_k = \lambda_k \mathbf{e}_k, \quad \mathbf{e}_i \cdot \mathbf{e}_k = \delta_{ik}, \quad (i, k = 1, 2, \ldots, n)$$

and for an arbitrary vector **x** in the orthonormal base $\mathbf{e}_1, \ldots, \mathbf{e}_n$

$$\mathbf{x} = \sum_{k=1}^{n} x_k \mathbf{e}_k$$

then

$$\mathbf{S}\mathbf{x} = \mathbf{S} \sum_{k=1}^{n} x_k \mathbf{e}_k = \sum_{k=1}^{n} x_k \mathbf{S}\mathbf{e}_k = \sum_{k=1}^{n} (\lambda_k x_k) \mathbf{e}_k . \tag{23.9}$$

S is said to act in the space by dilating the components of each vector in n mutually orthogonal directions in the ratios $|\lambda_1|$, $|\lambda_2| \ldots$, and $|\lambda_n|$ (if a λ_k is negative then the sense of the component is also reversed). This is the reason why a mapping performed by a symmetric operator is sometimes called (mostly in the three-dimensional ordinary space) a *dilatation* and plays an important role in elasticity theory.

24. Quadratic Forms

The Main Diagonal Theorem plays an important role in the theory and classification of quadratic forms. Let a_{ik} be real constants ($i, k = 1, 2, \ldots, n$); the expression

$$Q(x) = \sum_{i,k=1}^{n} a_{ik} x_i x_k \tag{24.1}$$

is called a *quadratic form* of the real variables $x_1, x_2, \ldots, x_n$. The nth-order matrix $A = [a_{ik}]$ is called the *coefficient matrix* and is always considered to be symmetric. If a_{ik} were not equal to a_{ki}, the following change of notation could be made in (24.1):

$$a_{ik} x_i x_k + a_{ki} x_k x_i = (a_{ik} + a_{ki}) x_i x_k$$

$$= \frac{a_{ik} + a_{ki}}{2} x_i x_k + \frac{a_{ik} + a_{ki}}{2} x_k x_i = a_{ik} x_i x_k + \tilde{a}_{ki} x_k x$$

where $\tilde{a}_{ik} = \tilde{a}_{ki}$. Introducing the column vector $x = \operatorname{col}[x_1, x_2, \ldots, x_n]$ of the variables, (24.1) can be put in the form

$$Q(x) = x' A x , \tag{24.2}$$

where $A = A'$. The symmetric matrix A can be considered a coordinate matrix of a linear operator **A** in some orthonormal base where x is the coordinate column vector of an arbitrary vector **x** in $\mathcal{E}^n(\mathcal{R})$. Thus, in this base (24.2) is the scalar product of **x** and **Ax** (see (19.11)).

A variant of the Main Diagonal Theorem concerning quadratic forms is

Theorem 24.1. *(Principal Axis Theorem).* For an arbitrary quadratic form (24.1) (or (24.2)) an orthogonal coordinate transformation can be found such that (24.1) assumes its 'canonical form' in the new coordinates; that is, an orthogonal matrix M can be found such that, performing the coordinate transformation

$$\tilde{x} = M'x \tag{24.3}$$

(see (20.10)), we have

$$\sum_{i,k=1}^{n} a_{ik}\ x_i x_k \equiv \sum_{i=1}^{n} \lambda_i \tilde{x}_i^2 ; \tag{24.4}$$

here λ_i and the ith column vector m_i of M is an eigenvalue, and a corresponding eigenvector, of the symmetric matrix $A(i=1, 2, \ldots, n)$ and $\lambda_1, \ldots, \lambda_n$ are all the eigenvalues of A taken according to their respective multiplicities.

Proof. This is an immediate consequence of Corollary 23.12, since if the coordinate transformation (24.3) is performed with the orthogonal matrix M occurring in this corollary we obtain, from (24.2),

$$x' Ax = (M\tilde{x})' AM\tilde{x} = \tilde{x}' M' AM\tilde{x}$$

$$= \tilde{x}' D\tilde{x} = \sum_{i=1}^{n} \tilde{x}_i \sum_{k=1}^{n} \lambda_i\, \delta_{ik}\, \tilde{x}_k = \sum_{i=1}^{n} \lambda_i \tilde{x}_i^2 \,.\,!$$

Definition 24.1. The quadratic form (24.2) is said to be *positive definite (negative definite)* if for $x \neq 0$

$$x' Ax \geq 0 \quad (x' Ax \leq 0)\,.$$

Definition 24.2. The quadratic form (24.2) is said to be *positive semi-definite (negative semi-definite)* if for all x

$$x' Ax \geq 0 \quad (x' Ax \leq 0)\,.$$

Definition 24.3. The quadratic form (24.2) is said to be *indefinite* if it assumes both positive and negative values (i.e. it is neither positive nor negative semi-definite).

A symmetric matrix is said to be definite, semi-definite, or indefinite according to whether it is the coefficient matrix of a definite, a semi-definite, or an indefinite quadratic form, respectively.

Theorem 24.2. The quadratic form (24.2) is a) positive (negative) definite, b) positive (negative) semi-definite, c) indefinite, if and only if a) all eigenvalues of A are positive (negative), b) all eigenvalues are non-negative (non-positive), c) A has both positive and negative eigenvalues, respectively.

Proof. This is an immediate consequence of (24.4). !

The classification of quadratic forms according to Definitions 24.1–24.3 is applied in the classification of conic sections in the following.

Example 24.1. Consider a Cartesian orthogonal system of coordinates x_1, x_2 defined by the origin and the orthonormal base $\mathbf{e}_1$, $\mathbf{e}_2$ in the ordinary plane. The locus of points (x_1, x_2) that satisfy the equation

$$a_{11} x_1^2 + 2a_{12} x_1 x_2 + a_{22} x_2^2 + b_1 x_1 + b_2 x_2 + c = 0 \tag{24.5}$$

is called a *quadratic curve* (or a *conic section*). Here a_{ik}, b_i, and c are real constants ($i = 1, 2$; $k = 1, 2$). Introducing the notation $a_{21} = a_{12}$ and

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

(24.5) assumes the form

$$x' A x + b' x + c = 0. \tag{24.6}$$

Denote by λ_1 and λ_2 the two eigenvalues of the matrix A (these may well be equal), by

$$\mathbf{m}_1 = m_{11} \mathbf{e}_1 + m_{21} \mathbf{e}_2, \quad \mathbf{m}_2 = m_{12} \mathbf{e}_1 + m_{22} \mathbf{e}_2$$

two corresponding orthogonal unit eigenvectors and let

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}.$$

Then by Theorem 24.1 the coordinate transformation (24.3) turns the quadratic part of (24.6) into the form $\lambda_1 \tilde{x}_1^2 + \lambda_2 \tilde{x}_2^2$. Thus, in the Cartesian orthogonal coordinate system determined by the same origin O and by the orthonormal base $\mathbf{m}_1$, $\mathbf{m}_2$, the equation of our quadratic form is

$$\lambda_1 \tilde{x}_1^2 + \lambda_2 \tilde{x}_2^2 + b' M \tilde{x} + c = 0 .$$

Since $b' M = [b_1 m_{11} + b_2 m_{21},\ b_1 m_{12} + b_2 m_{22}] = [\tilde{b}_1, \tilde{b}_2]$, we have

$$\lambda_1 \tilde{x}_1^2 + \lambda_2 \tilde{x}_2^2 + \tilde{b}_1 \tilde{x}_1 + \tilde{b}_2 \tilde{x}_2 + c = 0. \tag{24.7}$$

Case a). If $\lambda_1 \lambda_2 = \det A > 0$, i.e. $x' A x$ is definite, then by a simple displacement of the origin performed by the introduction of the new coordinates

$$\hat{x}_1 = \tilde{x}_1 + \frac{\tilde{b}_1}{2\lambda_1}, \qquad \hat{x}_2 = \tilde{x}_2 + \frac{\tilde{b}_2}{2\lambda_2}$$

(24.7) assumes the form

$$\lambda_1 \hat{x}_1^2 + \lambda_2 \hat{x}_2^2 = d$$

where

$$d = \lambda_1 \left(\frac{\tilde{b}_1}{2\lambda_1}\right)^2 + \lambda_2 \left(\frac{\tilde{b}_2}{2\lambda_2}\right)^2 - c.$$

Depending on whether the sign of d is equal or opposite to the common sign of λ_1, λ_2 our conic section is an

$$\textit{ellipse}: \; \frac{\hat{x}_1^2}{\alpha_1^2} + \frac{\hat{x}_2^2}{\alpha_2^2} = 1,$$

or an

$$\textit{imaginary ellipse}: \; \frac{\hat{x}_1^2}{\alpha_1^2} + \frac{\hat{x}_2^2}{\alpha_2^2} = -1,$$

(the latter has no point; in both cases

$$\alpha_1^2 = \left|\frac{d}{\lambda_1}\right|, \qquad \alpha_2^2 = \left|\frac{d}{\lambda_2}\right|).$$

If, finally, $d = 0$, our conic section is a

$$\textit{point}: \; \frac{\hat{x}_1^2}{\beta_1^2} + \frac{\hat{x}_2^2}{\beta_2^2} = 0$$

where $\beta_1^2 = 1/|\lambda_1|$, $\beta_2^2 = 1/|\lambda_2|$.

Case b). If $\lambda_1\lambda_2 = \det A = 0$, i.e. $x'Ax$ is semi-definite, then we may exclude the case $\lambda_1 = \lambda_2 = 0$ as of no interest, since this implies that A is the zero matrix and (24.5) is not quadratic. If, say $\lambda_2 \neq 0$, then by a simple displacement of the origin a new Cartesian orthonormal coordinate system $\hat{x}_1$, $\hat{x}_2$ can be introduced in which it can be seen that our conic section is a

$$\textit{parabola}: \; \hat{x}_2^2 = 2p\hat{x}_1, \; p \neq 0,$$

or a

$$\textit{pair of parallel lines}: \; \hat{x}_2^2 = \alpha^2, \qquad \alpha \neq 0,$$

or an

$$\textit{imaginary pair of lines}: \; \hat{x}_2^2 = -\alpha^2, \; \alpha \neq 0, \; \text{(i.e. an empty set)},$$

or a

$$\textit{line}: \; \hat{x}_2^2 = 0.$$

Case c). If $\lambda_1\lambda_2 = \det A \leq 0$, i.e. $x'Ax$ is indefinite, then again by a displacement of the origin the equation of the conic section can be written in its simplest form and we have either a

$$\textit{hyperbola}: \; \frac{\hat{x}_1^2}{\alpha_1^2} - \frac{\hat{x}_2^2}{\alpha_2^2} = 1$$

or a

$$\textit{pair of intersecting lines}: \; \frac{\hat{x}_1^2}{\alpha_1^2} - \frac{\hat{x}_2^2}{\alpha_2^2} = 0,$$

i.e.

$$\left(\frac{\hat{x}_1}{\alpha_1} + \frac{\hat{x}_2}{\alpha_2}\right)\left(\frac{\hat{x}_1}{\alpha_1} - \frac{\hat{x}_2}{\alpha_2}\right) = 0.$$

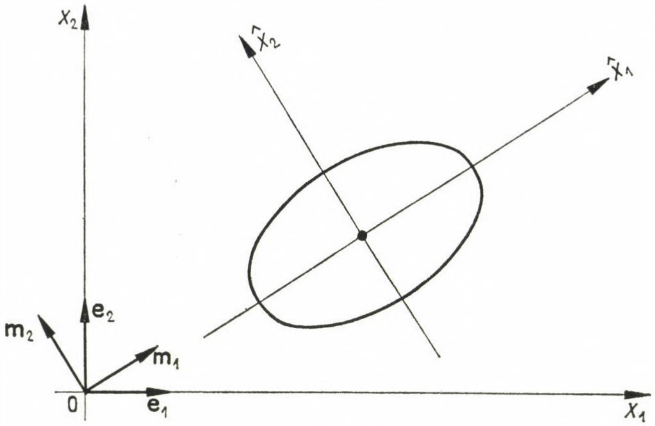

x_2
$\hat{x}_2$
$\hat{x}_1$
$\mathbf{e}_2$
$\mathbf{m}_2$
$\mathbf{m}_1$
0
$\mathbf{e}_1$
x_1

Fig. **24.1(a)**

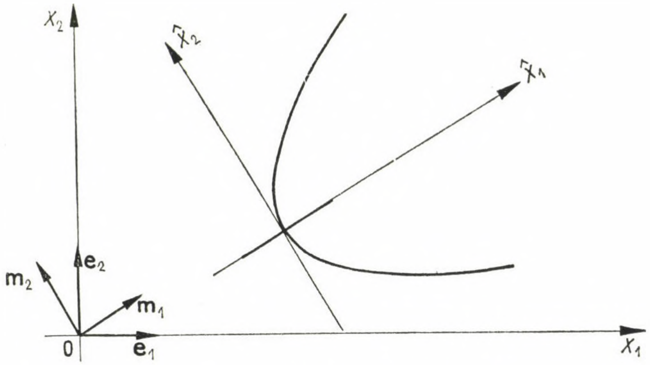

x_2
$\hat{x}_2$
$\hat{x}_1$
$\mathbf{e}_2$
$\mathbf{m}_2$
$\mathbf{m}_1$
0
$\mathbf{e}_1$
x_1

Fig. 24.1(b)

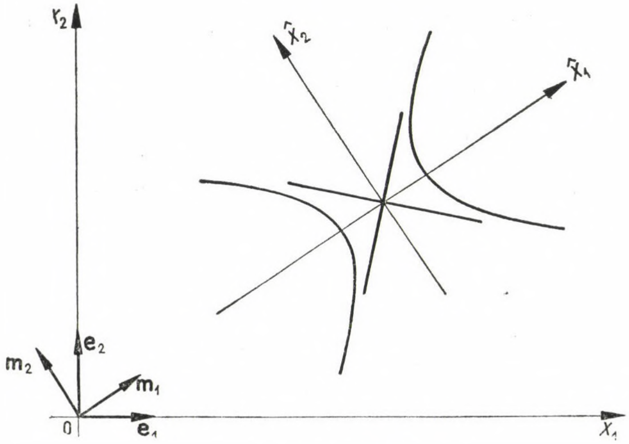

x_2
$\hat{x}_2$
$\hat{x}_1$
$\mathbf{e}_2$
$\mathbf{m}_2$
$\mathbf{m}_1$
0
$\mathbf{e}_1$
x_1

Fig. **24.1(c)**

As we have seen in all cases, the new coordinate system in which the equation has been put into its simplest form had axes coinciding with the principal axes of the conic section (Fig. 24.1 a, b, c). Hence the name Principal Axis Theorem.

25. Decomposition of Operators

We consider first the almost trivial question of decomposing a linear operator into the sum of a symmetric and a skew-symmetric operator.

THEOREM 25.1. An arbitrary linear operator $\mathbf{A}$ of $\mathcal{E}^n(\mathcal{R})$ has a uniquely determined decomposition into the sum of a symmetric and a skew-symmetric operator, i.e. there is one and only one symmetric operator $\mathbf{S}$ and skew-symmetric operator $\mathbf{Z}$ such that

$$\mathbf{A} = \mathbf{S} + \mathbf{Z}. \tag{25.1}$$

Proof. First the uniqueness of the decomposition (25.1) will be proved. Assume that (25.1) holds and

$$\mathbf{S}^* = \mathbf{S}, \quad \mathbf{Z}^* = -\mathbf{Z}$$

(see Definitions 23.1–23.3). Take the adjoint of both sides of (25.1) (it follows from the definition that the adjoint of a sum is the sum of the adjoints)

$$\mathbf{A}^* = \mathbf{S}^* + \mathbf{Z}^* = \mathbf{S} - \mathbf{Z}.$$

Adding the last equality to (25.1) we get

$$\mathbf{A} + \mathbf{A}^* = 2\mathbf{S};$$

subtracting it from (25.1) we get

$$\mathbf{A} - \mathbf{A}^* = 2\mathbf{Z},$$

that is,

$$\mathbf{S} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^*), \qquad \mathbf{Z} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^*) \tag{25.2}$$

and hence $\mathbf{S}$ and $\mathbf{Z}$ are uniquely determined. Adding the two equalities of (25.2) we get (25.1). At the same time, it is obvious that $\mathbf{S}$ and $\mathbf{Z}$ as defined by (25.2) are symmetric and skew-symmetric, respectively. !

Applying this theorem to the matrix of the operator we get

COROLLARY 25.2. For an arbitrary real quadratic matrix $A = [a_{ik}]$ of order n there is one and only one symmetric matrix $S = [s_{ik}]$ and skew-symmetric matrix $Z = [z_{ik}]$ such that

$$A = S + Z$$

and

$$s_{ik} = \frac{1}{2}(a_{ik} + a_{ki}), \quad z_{ik} = \frac{1}{2}(a_{ik} - a_{ki}) \quad (i, k = 1, 2, \ldots, n). \tag{25.3}$$

An important special type of operator is formed by those that preserve the scalar product of vectors and, as a consequence, the metric relations in the space.

Definition 25.1. The linear operator $\mathbf{M}$ of $\mathscr{E}^n(\mathscr{R})$ is said to be *isometric* if for all $\mathbf{x},\ \mathbf{y} \in \mathscr{E}^n(\mathscr{R})$

$$(\mathbf{Mx}) \cdot (\mathbf{My}) = \mathbf{x} \cdot \mathbf{y}. \tag{25.4}$$

It follows from this definition at once that an isometric operator preserves the norm of the vector, i.e. for all $\mathbf{x} \in \mathscr{E}^n(\mathscr{R})$

$$\| \mathbf{Mx} \| = \| \mathbf{x} \|. \tag{25.5}$$

Note that isometric operators could have been defined by (25.5), since it can be proved that the latter condition implies (25.4). Conditions (25.4) and (25.5) imply that an isometric operator preserves the angle of vectors (see (18.11)) and, as a consequence, orthogonality.

Theorem 25.3. The linear operator $\mathbf{M}$ of $\mathscr{E}^n(\mathscr{R})$ is isometric if and only if

$$\mathbf{M}^*\mathbf{M} = \mathbf{I}. \tag{25.6}$$

Proof. If $\mathbf{M}$ is isometric then for all $\mathbf{x}, \mathbf{y} \in \mathscr{E}^n(\mathscr{R})$

$$\mathbf{x} \cdot \mathbf{y} = (\mathbf{Mx}) \cdot (\mathbf{My}) = \mathbf{x} \cdot (\mathbf{M}^*\mathbf{My})$$

(see Definition 23.1 of the adjoint), that is,

$$\mathbf{x} \cdot (\mathbf{y} - \mathbf{M}^*\mathbf{My}) = 0.$$

Since this holds with a fixed $\mathbf{y}$ for all $\mathbf{x}$, it follows that $\mathbf{y} - \mathbf{M}^*\mathbf{My}$ is the zero vector (see Lemma 19.3). Hence, for all $\mathbf{y} \in \mathscr{E}^n(\mathscr{R})$

$$\mathbf{M}^*\mathbf{My} = \mathbf{y},$$

and this means that $\mathbf{M}^*\mathbf{M} = \mathbf{I}$. Conversely, if $\mathbf{M}^*\mathbf{M} = \mathbf{I}$ then, for all $\mathbf{x}, \mathbf{y} \in \mathscr{E}^n(\mathscr{R}), \mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot (\mathbf{Iy}) = \mathbf{x} \cdot (\mathbf{M}^*\mathbf{My}) = (\mathbf{Mx}) \cdot (\mathbf{My})$. !

Theorem 25.4. The linear operator $\mathbf{M}$ of $\mathscr{E}^n(\mathscr{R})$ is isometric if and only if in an arbitrary orthonormal base its coordinate matrix is orthogonal.

Proof. If in an orthonormal base the coordinate matrix of the operator $\mathbf{M}$ is M, i.e. $\mathbf{M} \simeq M$, then by Theorem 23.2 $\mathbf{M}^* \simeq M'$. Hence, (25.6) implies $M'M = \mathrm{I}$ and conversely (see Definition 20.1). !

From properties (20.7) and (20.8) of orthogonal matrices, taking into account (21.18) and Theorem 21.3, we get

Corollary 25.5. The linear operator $\mathbf{M}$ of $\mathscr{E}^n(\mathscr{R})$ is isometric if and only if it is non-degenerate and

$$\mathbf{M}^* = \mathbf{M}^{-1}.$$

COROLLARY 25.6. If $\mathbf{M}$ is isometric then the adjoint $\mathbf{M}^*$ is also isometric and

$$\mathbf{M}\mathbf{M}^* = \mathbf{I}. \tag{25.7}$$

The rest of this Section deals with the possibility of the decomposition of a linear operator into the product of a symmetric and an isometric operator. The importance of such a decomposition will be shown in Example 25.1. We need some lemmas concerning the following:

DEFINITION 25.2. The linear operator $\mathbf{B}$ of $\mathscr{E}^n(\mathscr{R})$ is said to be *positive definite* if it is symmetric and for all $\mathbf{x} \in \mathscr{E}^n(\mathscr{R})$, $\mathbf{x} \neq \mathbf{0}$, satisfies the inequality

$$\mathbf{x} \cdot (\mathbf{B}\mathbf{x}) > 0.$$

LEMMA 25.7. The symmetric linear operator $\mathbf{B}$ is positive definite if and only if in an arbitrary orthonormal base where $\mathbf{B} \simeq B$, $\mathbf{x} \simeq x$ the quadratic form

$$x' Bx = \mathbf{x} \cdot (\mathbf{B}\mathbf{x})$$

is positive definite.

Proof. This is an immediate consequence of Definition 24.1 and (19.11). !

LEMMA 25.8. A symmetric operator is positive definite if and only if all its eigenvalues are positive.

Proof. This is an immediate consequence of Theorem 24.2 and the previous lemma. !

LEMMA 25.9. If $\mathbf{A}$ is a non-degenerate linear operator in $\mathscr{E}^n(\mathscr{R})$ then both $\mathbf{A}\mathbf{A}^*$ and $\mathbf{A}^*\mathbf{A}$ are positive definite operators.

Proof. The proposition concerning $\mathbf{A}\mathbf{A}^*$ will be proved; the other one can be proved similarly. By (23.2)

$$(\mathbf{A}\mathbf{A}^*)^* = (\mathbf{A}^*)^*\mathbf{A}^* = \mathbf{A}\mathbf{A}^*$$

i.e. $\mathbf{A}\mathbf{A}^*$ is symmetric. From the non-degeneracy of $\mathbf{A}$ it follows that $\mathbf{A}^*$ is non-degenerate and hence if $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{A}^*\mathbf{x} \neq \mathbf{0}$. Thus for all $\mathbf{x} \in \mathscr{E}^n(\mathscr{R})$, $\mathbf{x} \neq \mathbf{0}$, we have

$$\mathbf{x} \cdot (\mathbf{A}\mathbf{A}^*\mathbf{x}) = (\mathbf{A}^*\mathbf{x}) \cdot (\mathbf{A}^*\mathbf{x}) = \| \mathbf{A}^*\mathbf{x} \|^2 > 0. \ !$$

LEMMA 25.10. For an arbitrary positive definite operator $\mathbf{B}$ there can be found one and only one positive definite operator $\mathbf{S}$ such that

$$\mathbf{B} = \mathbf{S}^2 = \mathbf{S}\mathbf{S};$$

$\mathbf{S}$ is then called the square root of $\mathbf{B}$ and is denoted by $\mathbf{S} = \sqrt{\mathbf{B}}$.

Proof. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be an orthonormal base formed by eigenvectors of $\mathbf{B}$, and $\mathbf{B}\mathbf{e}_k = \lambda_k\mathbf{e}_k$ $(k = 1, 2, \ldots, n)$. It follows from Lemma 25.8 that $\lambda_k > 0$ $(k = 1, 2, \ldots, n)$. Define the linear operator $\mathbf{S}$ (by prescribing the vectors into which the base vectors are mapped) as follows:

$$\mathbf{S}\mathbf{e}_k = \sqrt{\lambda_k}\mathbf{e}_k \quad (k = 1, 2, \ldots, n).$$

Since $\sqrt{\lambda_k} > 0$ $(k = 1, 2, \ldots, n)$, it follows from Corollary 23.11 and Lemma 25.8 that $\mathbf{S}$ is a positive definite operator. At the same time

$$\mathbf{S}^2 \mathbf{e}_k = \mathbf{S}(\mathbf{S}\mathbf{e}_k) = \mathbf{S}(\sqrt{\lambda_k}\,\mathbf{e}_k) = \sqrt{\lambda_k}\sqrt{\lambda_k}\,\mathbf{e}_k = \lambda_k\mathbf{e}_k = \mathbf{B}\mathbf{e}_k, \quad (k = 1, ,2 \ldots, n);$$

hence $\mathbf{S}^2 = \mathbf{B}$.

We show that $\mathbf{S}$ is determined uniquely. Assume that $\mathbf{V}$ is a positive definite operator and $\mathbf{V}^2 = \mathbf{B}$. Denote by $\mathbf{a}_1, \ldots, \mathbf{a}_n$ an orthonormal base formed by eigenvectors of $\mathbf{V}$ and let $\mathbf{V}\mathbf{a}_j = \mu_j\mathbf{a}_j$ $(j = 1, 2, \ldots, n)$. Then

$$\mathbf{B}\mathbf{a}_j = \mathbf{V}^2\mathbf{a}_j = \mathbf{V}(\mathbf{V}\mathbf{a}_j) = \mathbf{V}(\mu_j\mathbf{a}_j) = \mu_j^2\,\mathbf{a}_j.$$

Thus $\mathbf{a}_j$ is an eigenvector and μ_j^2 a corresponding eigenvalue of $\mathbf{B}$ $(j = 1, 2, \ldots, n)$. If the indices are chosen appropriately we may write $\mu_j^2 = \lambda_j$ $(j = 1, 2, \ldots, n)$. Since $\mathbf{V}$ is positive definite, $\mu_j > 0$; thus $\mu_j = \sqrt{\lambda_j}$. Let

$$\mathbf{e}_k = \sum_{i=1}^{n} m_{ik}\,\mathbf{a}_i \qquad (k = 1, 2, \ldots, n).$$

However, according to Theorem 23.7 $\mathbf{e}_k$ is orthogonal to those eigenvectors $\mathbf{a}_i$ of $\mathbf{B}$ that belong to eigenvalues different from λ_k. If the multiplicity of λ_k is p, $\lambda_{i_1} = \lambda_{i_2} = \ldots = \lambda_{i_p} = \lambda_k$ and $\lambda_i \neq \lambda_k$ $(i \neq i_1, i_2, \ldots, i_p)$ then

$$m_{ik} = \mathbf{a}_i \cdot \mathbf{e}_k = 0 \text{ for } i \neq i_1, i_2, \ldots, i_p,$$

and thus

$$\mathbf{e}_k = m_{i_1 k}\,\mathbf{a}_{i_1} + m_{i_2 k}\,\mathbf{a}_{i_2} + \ldots + m_{i_p k}\,\mathbf{a}_{i_p}.$$

It follows from

$$\mathbf{V}\mathbf{a}_{i_h} = \sqrt{\lambda_{i_h}}\,\mathbf{a}_{i_h} = \sqrt{\lambda_k}\,\mathbf{a}_{i_h}, \qquad (h = 1, 2, \ldots, p)$$

that

$$\begin{aligned}\mathbf{V}\mathbf{e}_k &= \mathbf{V}(m_{i_1 k}\,\mathbf{a}_{i_1} + \ldots + m_{i_p k}\,\mathbf{a}_{i_p})\\ &= m_{i_1 k}\sqrt{\lambda_k}\,\mathbf{a}_{i_1} + \ldots + m_{i_p k}\sqrt{\lambda_k}\,\mathbf{a}_{i_p} = \sqrt{\lambda_k}\,\mathbf{e}_k\,;\end{aligned}$$

thus $\mathbf{V}\mathbf{e}_k = \mathbf{S}\mathbf{e}_k$ $(k = 1, 2, \ldots, n)$, and hence $\mathbf{V} = \mathbf{S}$. !

THEOREM 25.11. If $\mathbf{A}$ is a non-degenerate operator in $\mathcal{E}^n(\mathfrak{R})$ then there is one and only one positive definite (symmetric) operator $\mathbf{S}$ and isometric operator $\mathbf{M}$ such that

$$\mathbf{A} = \mathbf{S}\mathbf{M} \tag{25.8}$$

and there is one and only one positive definite operator $\mathbf{S}_1$ and isometric operator $\mathbf{M}_1$ such that

$$\mathbf{A} = \mathbf{M}_1 \mathbf{S}_1. \tag{25.9}$$

Proof. The uniqueness of decomposition (25.8) will be proved first. If (25.8) holds for some positive definite $\mathbf{S}$ and isometric $\mathbf{M}$ then

$$\mathbf{AA}^* = \mathbf{SM}(\mathbf{SM})^* = \mathbf{SMM}^*\mathbf{S} = \mathbf{S}^2$$

where (23.2) and (25.7) have been applied. Since, by Lemma 25.9, $\mathbf{AA}^*$ is positive definite, $\mathbf{S} = \sqrt{\mathbf{AA}^*}$ is uniquely determined according to Lemma 25.10. The positive definite operator $\mathbf{S}$ is, obviously, non-degenerate; thus $\mathbf{M}$ is determined by (25.8) uniquely:

$$\mathbf{M} = \mathbf{S}^{-1}\mathbf{A}.$$

Conversely, if $\mathbf{S}$ and $\mathbf{M}$ are defined by $\mathbf{S} = \sqrt{\mathbf{AA}^*}$ and $\mathbf{M} = \mathbf{S}^{-1}\mathbf{A}$, respectively, then (25.8) holds; $\mathbf{S}$ is positive definite and only the isometry of $\mathbf{M}$ remains to be proved. But

$$\mathbf{MM}^* = \mathbf{S}^{-1}\mathbf{A}(\mathbf{S}^{-1}\mathbf{A})^* = \mathbf{S}^{-1}\mathbf{AA}^*\mathbf{S}^{-1*} = \mathbf{S}^{-1}\mathbf{S}^2\mathbf{S}^{-1} = \mathbf{I}$$

where we have taken into account that the inverse of a non-degenerate symmetric operator is obviously also symmetric. Thus $\mathbf{M}$ is isometric and the proposition concerning (25.8) has been completely proved.

The decomposition (25.9) can be proved by applying to $\mathbf{A}^*$ what has already been proved. That is to say for the non-degenerate operator $\mathbf{A}^*$ there is a positive definite $\mathbf{S}_1$ and an isometric $\mathbf{M}_0$ such that $\mathbf{A}^* = \mathbf{S}_1\mathbf{M}_0$ where $\mathbf{S}_1 = \sqrt{\mathbf{A}^*(\mathbf{A}^*)^*} = \sqrt{\mathbf{A}^*\mathbf{A}}$. According to Corollary 25.6 $\mathbf{M}_1 = \mathbf{M}_0^*$ is isometric and

$$\mathbf{A} = (\mathbf{A}^*)^* = (\mathbf{S}_1\mathbf{M}_0)^* = \mathbf{M}_0^*\mathbf{S}_1^* = \mathbf{M}_1\mathbf{S}_1. \ !$$

EXAMPLE 25.1. Consider the ordinary (two-dimensional Euclidean) plane or the ordinary (three-dimensional Euclidean) space. An isometric operator maps the plane or the space onto itself, preserving the length (norm) and the angle of vectors. If we fix a point O, the origin, an isometric operator moves the position vectors, i.e. moves the points so that the distance between points remains unaltered and the point O is mapped into itself. Thus the plane or the space is mapped onto itself, as if it were a solid fixed in the point O and had been rotated around this point (following the rotation a reflection of the points may also be needed). Therefore we may say that an isometric mapping of the position vectors of the plane or the space is a rotation (plus possibly a reflection). According to Theorem 25.11, any non-degenerate linear mapping of the ordinary plane or space is a rotation (and possibly a reflection) followed by a dilatation (with positive dilatation coefficients; see the end of Section 23).

Exercises

VI.1 A Cartesian orthogonal coordinate system x_1, x_2 is introduced in the ordinary plane corresponding to the orthonormal base $\mathbf{e}_1, \mathbf{e}_2$. Find the coordinate matrix of the linear operator $\mathbf{A}$ and the coordinate matrix of its inverse if $\mathbf{A}$ acts on the position vectors in the following way:

a) reflects the points in the axis x_1;
b) reflects the points in the axis x_2;
c) reflects the points in the origin;
d) reflects the points in the line $x_2 = x_1$.

VI.2 The linear operators $\mathbf{A}$ and $\mathbf{B}$ of the ordinary plane rotate the position vectors around the origin in the positive direction by the angles $\pi/4$ and $\pi/6$, respectively. In an orthonormal base find the coordinate matrix of $\mathbf{A}$, $\mathbf{B}$ and of the operator that rotates the position vectors in the positive direction by the angle $5\pi/12 = \pi/4 + \pi/6$.

VI.3 Find the coordinate matrix of the linear operator $\mathbf{A}$ of the ordinary plane in an orthonormal base if $\mathbf{A}$ rotates the position vectors around the origin by the angle $2\pi/3$ and makes them eight times as long as before.

VI.4 The origin is fixed and a dextral orthonormal base $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is introduced in the ordinary space. Find the coordinate matrix of the linear operator

a) $\mathbf{A}_i$ that reflects the position vectors in the plane spanned by $\mathbf{e}_k$ and $\mathbf{e}_l$ and passing through the origin ($i \neq k$, $i \neq l$, $k \neq l$, $i = 1, 2, 3$).
b) $\mathbf{B}_i$ that rotates the position vectors around $\mathbf{e}_i$ in the positive direction by the angle π ($i = 1, 2, 3$).

VI.5 Let

$$\tilde{\mathbf{e}}_1 = \frac{1}{\sqrt{6}}\mathbf{e}_1 + \frac{2}{\sqrt{6}}\mathbf{e}_2 + \frac{-1}{\sqrt{6}}\mathbf{e}_3$$

$$\tilde{\mathbf{e}}_2 = \frac{1}{\sqrt{2}}\mathbf{e}_1 \qquad + \frac{1}{\sqrt{2}}\mathbf{e}_3$$

$$\tilde{\mathbf{e}}_3 = \frac{1}{\sqrt{3}}\mathbf{e}_1 + \frac{-1}{\sqrt{3}}\mathbf{e}_2 + \frac{-1}{\sqrt{3}}\mathbf{e}_3$$

be a new base in the ordinary space. Find the coordinate matrices of the linear operators $\mathbf{A}_1$ and $\mathbf{B}_1$ of Exercise VI.4 in this new base.

VI.6 Show that if a linear operator $\mathbf{A}$ of a linear space satisfies $\mathbf{A}^2 = \mathbf{A}$ then $\mathbf{A}$ has no eigenvalues different from 0 or 1.

VI.7 The quadratic matrix A of order n is called *nilpotent* if a natural number m exists such that $A^m = O$.

a) Show that if A is nilpotent then its only eigenvalue is zero.
b) Show that the nth-order matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

is nilpotent and has one and only one (linearly independent) eigenvector.

VI.8 Find the eigenvalues and eigenvectors of the following matrices:

$$\text{a)}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \qquad \text{b)}\begin{bmatrix} 1 & 0 & -2 \\ -1 & 1 & 0 \\ 0 & -2 & 4 \end{bmatrix}$$

$$\text{c)}\begin{bmatrix} \frac{11}{6} & \frac{1}{6} & \frac{2}{6} \\ \frac{1}{6} & \frac{11}{6} & -\frac{2}{6} \\ \frac{2}{6} & -\frac{2}{6} & \frac{8}{6} \end{bmatrix} \qquad \text{d)}\begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}.$$

VI.9 Assume that the linear operator **A** of the three-dimensional real Euclidea space $\mathcal{E}^3(\mathfrak{R})$ has the coordinate matrix

$$\mathbf{A} \simeq \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

in a base $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. Determine a one-dimensional and a two-dimensiona invariant subspace of **A**.

VI.10 Show that if **A** is a hermitian operator of $\mathcal{E}^n(\mathfrak{C})$ then all its eigenvalues are real.

VI.11 Show that if **A** is a skew-hermitian linear operator of $\mathcal{E}^n(\mathfrak{C})$ then all its eigenvalues are imaginary.

VI.12 Transform

$$x_1^2 + 3x_2^2 + 2x_3^2 + 4x_4^2 + 2x_1x_2$$

by an orthogonal transformation into main diagonal form.

VI.13 Show that if **A** is a regular linear operator in a linear space and $\mathbf{A}^{-1} = \mathbf{A}$ then **A** cannot have eigenvalues different from ±1.

VI.14 Show that if **A** is a regular linear operator in a linear space, and $\mathbf{A}^{-1} = -\mathbf{A}$, then **A** cannot have eigenvalues different from $\pm i$.

VI.15 Assume that in an orthonormal base the coordinate matrix of the regular linear operator **A** of the two-dimensional real Euclidean space $\mathcal{E}^2(\mathfrak{R})$ is

$$\mathbf{A} \simeq \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}.$$

a) Determine in the same base the coordinate matrices of the isometric operator $\mathbf{M}_1$ and the (symmetric) positive definite operator $\mathbf{S}_1$ in the decomposition $\mathbf{A} = \mathbf{M}_1\mathbf{S}_1$.

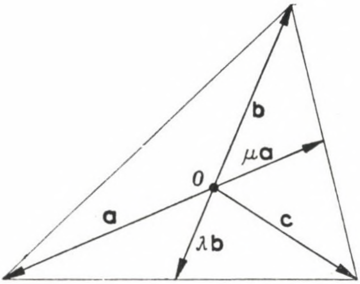
b) Determine in the same base the coordinate matrices of the (symmetric) positive definite operator **S** and the isometric operator **M** in the decomposition $\mathbf{A} = \mathbf{SM}$.

ANSWERS AND HINTS TO EXERCISES

Exercises to Chapter One (page 39)

I.1 Hint: Use the position vectors of the vertices. For the ‘if’ part place the origin at the intersection point of the diagonals. For the ‘only if’ part place the origin at a vertex of the parallelogram.

I.2 Place the origin at the intersection point of any two medians. On the figure: $\mathbf{a} + \mathbf{c} = 2\lambda\mathbf{b}$, $\mathbf{b} + \mathbf{c} = 2\mu\mathbf{a}$. Hence $\mu = \lambda = -\frac{1}{2}$ and $\mathbf{c} = -(\mathbf{a} + \mathbf{b})$. Thus $\frac{1}{2}(\mathbf{a} + \mathbf{b}) = -\frac{1}{2}\mathbf{c}$.



I.3 Either $|\,\mathbf{a}\,| = |\,\mathbf{b}\,|$ or $\mathbf{a}$ is orthogonal to $\mathbf{b}$.

I.4 Hint: Let $\beta \neq 0$. Write the scalar product and divide it by β^2. Answer: $|\,\mathbf{a}\,| = |\,\mathbf{b}\,|$ and $\mathbf{a}$ is orthogonal to $\mathbf{b}$.

I.5 Their angle is $\pi/4$ or $3\pi/4$.

I.6 Place the origin at the intersection point of two altitudes. Denote the position vectors of the vertices of the triangle by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. Since the medians are perpendicular to the sides we have $\mathbf{b}(\mathbf{c} - \mathbf{a}) = 0$, $\mathbf{a}(\mathbf{c} - \mathbf{b}) = 0$. Hence $\mathbf{c}(\mathbf{b} - \mathbf{a}) = 0$.

I.7 Place the origin at one of the vertices. $A = \frac{1}{2}\,|\,\mathbf{a}\times\mathbf{b}\,|$. Introducing $a = |\,\mathbf{a}\,|$, $b = |\,\mathbf{b}\,|$ $c = |\,\mathbf{c}\,| = |\,\mathbf{b} - \mathbf{a}\,|$, we have

$$\begin{aligned}
16A^2 &= 4(\mathbf{a}\times\mathbf{b})\ (\mathbf{a}\times\mathbf{b}) = 4\mathbf{a}^2\mathbf{b}^2 - (2\mathbf{ab})^2 \\
&= 4\mathbf{a}^2\mathbf{b}^2 - [\mathbf{a}^2 + \mathbf{b}^2 - (\mathbf{a} - \mathbf{b})^2]^2 \\
&= (2ab + a^2 + b^2 - c^2)\ (2ab - a^2 - b^2 + c^2)
\end{aligned}$$

$$= [(a+b)^2 - c^2][c^2 - (a-b)^2]$$
$$= (a+b+c)(a+b-c)(c+a-b)(c-a+b)$$
$$= (a+b+c)(a+b+c-2c)(a+b+c-2b)(a+b+c-2a).$$

I.8 Hint: $\mathbf{x} = \sum_{i=1}^{3} \lambda_i \mathbf{a}_i,\ \lambda_i = \dfrac{\mathbf{r} \cdot \mathbf{a}_i}{|\mathbf{a}_i|^2}$ $(i = 1, 2, 3)$.

Answer: $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3$.

I.9 $\mathbf{v}_1 \simeq \left(\dfrac{-5}{6}, \dfrac{5}{6}, \dfrac{-10}{6}\right)$,

$\mathbf{v}_2 \simeq \left(\dfrac{23}{6}, \dfrac{19}{6}, \dfrac{-2}{6}\right)$.

I.10 If $\mathbf{x}$ is the position vector of P and $\mathbf{x}_1$, $\mathbf{x}_2$ are the position vectors of P_1 and P_2, then $\mathbf{x} = \dfrac{n\mathbf{x}_1 + m\mathbf{x}_2}{m+n}$.

I.11 The normal vector of the plane is $-6\mathbf{e}_1 + 2\mathbf{e}_2$. The equation is $-6(x_1 - 1) + 2(x_2 + 2) = 0$.

I.12 Hint: The normal vector of the plane is $\mathbf{n} = (\mathbf{a} - \mathbf{p}) \times \mathbf{v}$ where $\mathbf{v}$ is the direction vector of the line, $\mathbf{p}$ is the position vector of P, and $\mathbf{a}$ is the position vector of a point on the line.

Answer: $-12(x_1 - 3) + 16(x_2 - 4) = 0$.

I.13 $15(x_1 + 1) + 2(x_2 - 2) + 36(x_3 - 1) = 0$.

I.14 $8(x_1 + 1) - 14(x_2 + 1) + 5(x_3 + 7) = 0$.

I.15 $\dfrac{x_1 + 1}{19} = \dfrac{x_2 - 1}{2} = \dfrac{x_3 + 1}{31}$.

I.16 Hint: The direction vector of the line that intersects the given lines at right angles is $\mathbf{v} = \mathbf{v}_1 \times \mathbf{v}_2$ where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the direction vectors of the given lines. Choose a point on each line. If their respective position vectors are $\mathbf{x}_1$ and $\mathbf{x}_2$, the distance is

$$\left| (\mathbf{x}_2 - \mathbf{x}_1) \frac{\mathbf{v}_1 \times \mathbf{v}_2}{|\mathbf{v}_1 \times \mathbf{v}_2|} \right|.$$

Answer: $\dfrac{5}{3}$.

I.17 Hint: The normal transversal is the line of intersection of the planes S_1 and S_2, where S_1 contains one line and the normal transversal and S_2 contains the other line and the normal transversal. The normal vectors of S_1 and S_2 are $\mathbf{n}_1 = \mathbf{v}_1 \times (\mathbf{v}_1 \times \mathbf{v}_2)$ and $\mathbf{n}_2 = \mathbf{v}_2 \times (\mathbf{v}_1 \times \mathbf{v}_2)$, respectively, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the direction vectors of the given lines.

Answer: $\dfrac{8 - x_1}{4} = x_2 - 3 = x_3$.

I.18 $c = -50$.

I.19 $P(-2, 0, 0)$.

Exercises to Chapter Two (page 60)

II.1 Hint: Express z in trigonometric form.
Answer: $z_1 = \cos\,\pi/3 + i\,\sin\,\pi/3$, $z_2 = \cos\,(-\pi/3) + i\,\sin\,(-\pi/3)$, $z_3 = -1$.

II.2 Hint: $1 = |z_1|^2 = z_1 \cdot \bar{z}_1$.

II.3 $$(1 + iz)^n = e^{i\frac{\pi}{6}} \qquad z = -i\left(\sqrt[n]{e^{i\frac{\pi}{6}}} - 1\right)$$

z is real if $\cos\dfrac{\frac{\pi}{6} + 2k\pi}{n} = 1$.

But $\dfrac{\pi}{6} + 2k\pi \neq n \cdot 2l\pi$, $l = 0, \pm 1, \ldots$, $k = 0, 1, \ldots n - 1$.

II.4 Hint: Prove that $z^{n+3} = z^n$, $n = 1, 2, \ldots$.

II.5 $$\sum_{k=0}^{n-1} \omega_{nk}^2 = \sum_{k=0}^{n-1} \left[e^{i\frac{2k\pi}{n}}\right]^2 = \sum_{k=0}^{n-1} \left[e^{i\frac{4\pi}{n}}\right]^k = \frac{\left[e^{i\frac{4\pi}{n}}\right]^n - 1}{e^{i\frac{4\pi}{n}} - 1} = 0, \quad n \neq 2.$$

For $n = 2$, no.

II.6 $z_1 = 1 + 2i$, $z_2 = 1 + i$.

II.7 $z_k = 3 + 3i + \omega_{6k}$, $k = 0, 1, \ldots 5$.

II.8 Hint: The equation of any straight line or circle on the plane can be written in the form

$$A(x^2 + y^2) + Bx + Cy + D = 0$$

where A, B, C, D are real constants.
Apply formulae (5.23).

II.9 $u = 2$, $v = -4$.

II.10 Hint: Apply (7.9) and Corollary 7.9.

II.11 2.

II.12 a) $z^3 - (7 + i)z^2 + (15 + 6i)z - (9 + 9i)$

b) $z^4 - 8z^3 + 23z^2 - 30z + 18$.

Exercises to Chapter Three (page 104)

III.1 $$X = \begin{bmatrix} -2 & 0 \\ 4 & -1 \\ 9 & 9 \end{bmatrix}.$$

III.2 $$(A \cdot B)C = A(B \cdot C) = \begin{bmatrix} 14 \\ 11 \\ 1 \end{bmatrix}.$$

III.3 a) $A \cdot B = \begin{bmatrix} -11 & 10 \\ 11 & 6 \end{bmatrix} \neq \begin{bmatrix} 0 & 11 \\ 4 & -17 \end{bmatrix} = B \cdot A$

b) $AB + B(C + A) = \begin{bmatrix} -12 & 22 \\ 18 & -18 \end{bmatrix}$.

III.4 If $AB = BA$ then the identity holds.
In the example

$$AB = \begin{bmatrix} 9 & 2 & 6 \\ -5 & -9 & 9 \\ 2 & 34 & 2 \end{bmatrix}, \quad BA = \begin{bmatrix} -4 & 18 & 10 \\ 18 & 11 & 11 \\ 17 & -18 & 13 \end{bmatrix}$$

$AB \neq BA$.

III.5 At least the elements of the main diagonal are zeros.

III.6 In the first case there is no B different from zero. In the second case $B = \operatorname{col}[1, 1, -1]$.

III.7 b) If $AB = A$ and $BA = B$ then $(AB)A = A^2$ and $A(BA) = AB = A$. Using the associativity of matrix multiplication we get $A^2 = A$.

III.8 If $AB = BA$ and $A = A'$, $B = B'$ then $AB = B'A' = (AB)'$.

If $AB = (A \cdot B)'$ and $A = A'$, $B = B'$ then $AB = (A \cdot B)' = B'A' = BA$.

III.9 $I, \quad -I, \begin{bmatrix} \alpha & \beta \\ \dfrac{1-\alpha^2}{\beta} & -\alpha \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ \gamma & -1 \end{bmatrix}$

$\begin{bmatrix} -1 & 0 \\ \gamma & 1 \end{bmatrix}$, where α, $\beta \neq 0$, γ are real numbers.

III.10 If $z_2 = 0$ then $A = \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$ or $A = \begin{bmatrix} \pm 1 & 0 \\ 0 & \mp 1 \end{bmatrix}$. If $z_2 \neq 0$ then

$$A = \begin{bmatrix} \pm\sqrt{1 - z_2^2} & z_2 \\ z_2 & \mp\sqrt{1 - z_2^2} \end{bmatrix}.$$

III.11 $$\begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix} \begin{bmatrix} w_1 & w_2 \\ -\bar{w}_2 & \bar{w}_1 \end{bmatrix} = \begin{bmatrix} z_1 w_1 - z_2 \bar{w}_2 & z_1 w_2 + z_2 \bar{w}_1 \\ -(\bar{z}_1 \bar{w}_2 + \bar{z}_2 w_1) & \bar{z}_1 \bar{w}_1 - \bar{z}_2 w_2 \end{bmatrix}.$$

The product is not commutable. For example:

$$\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} 0 & w \\ -\bar{w} & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & w \\ -w & 0 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & \bar{z} \end{bmatrix}.$$

III.12 $B = [e_1, e_2, \ldots e_{n-1}\, b]$ where

$e_i = \operatorname{col}[0 \ldots 0, 1, 0 \ldots 0] \qquad (i = 1, 2, \ldots n - 1)$
$b_i = \operatorname{col}[c_1, c_2, \ldots c_{n-1}, 1]$.

III.13 No.
If $A = A^*$ then $A^2 = A \cdot A^* = [a_{ik}^{(2)}]$, and

$$a_{ii}^{(2)} = \sum_{l=1}^{n} a_{il} \overline{a}_{il} = \sum_{l=1}^{n} | a_{il} |^2 = 0$$

implies $a_{il} = 0 \quad (i, l = 1, 2, \ldots n)$.

III.14 Yes. For example:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

III.15 Hint: Use mathematical induction.

III.16 Suppose $a_1, \ldots, ca_k, \ldots a_n$ are linearly independent. From $\sum\limits_{l=1}^{n} \lambda_l a_l = 0$ follows

$$\sum_{\substack{l=1 \\ l \neq k}}^{n} \lambda_l a_l + \frac{\lambda_k}{c} ca_k = 0. \text{ Thus } \mu_l = 0,\ l \neq k \text{ and } \frac{\mu_k}{c} = 0.$$

Suppose $a_1, \ldots, a_k, \ldots a_n$ are linearly independent. From

$$\sum_{\substack{l=1 \\ l \neq k}}^{n} \lambda_l a_l + \lambda_k ca_k = 0 \text{ follows } \lambda_l = 0,\ l \neq k, \text{ and } \lambda_k c = 0, \text{ so } \lambda_k = 0.$$

III.17 Suppose $a_1, \ldots, a_l, \ldots, a_k, \ldots a_n$ are linearly independent. From

$$\sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j a_j + \lambda_k(a_k + ca_l) = 0$$

follows $\lambda_j = 0$, $j \neq l$, and $\lambda_l + c\lambda_k = 0$.
Hence also $\lambda_l = 0$. Suppose $a_1, \ldots, a_l, \ldots a_k + ca_l, \ldots a_n$ are linearly independent. From $\sum\limits_{j=1}^{n} \mu_j a_j = 0$ follows

$$\sum_{\substack{j=1 \\ j \neq k \\ j \neq l}}^{n} \mu_j a_j + \mu_k(a_k + ca_l) + (\mu_l - c\mu_k)\ a_l = 0.$$

Hence $\mu_l = 1$, $j \neq l$, and $\mu_l - c\mu_k = 0$, so $\mu_l = 0$.

III.18 2, since $a_3 = 2a_1 - \frac{1}{3} a_2$.

III.19 $a_3 = \text{col}\,[a,\ 0,\ 0]$, where $a \neq 0$.

III.20 $c = -2$ or $c = 1$.

III.21 a) 0. b) $(b + 2a)(b - a)^2$.

III.22 Hint: Add the first column to the others.

III.23 a) $x = \pm 1$, $x = \pm \sqrt{2}$.
b) $x = 1$, $x = 2$, $x = -1$.

III.24 Hint: Apply Theorem 10.7.

III.25 Hint: Apply Theorem 10.7.

III.26 Hint: Apply Theorem 10.7.

III.27 $$\begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{bmatrix} = 0.$$

III.28 $t \neq 0$.

III.29 $$A^{-1} = \begin{bmatrix} \frac{1}{a_1} & 0 & 0 & \dots & 0 \\ -\frac{a_2}{a_1} & 1 & 0 & \dots & 0 \\ -\frac{a_3}{a_1} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ -\frac{a_{n-1}}{a_1} & 0 & 0 & \dots & 0 \\ -\frac{a_n}{a_1} & 0 & 0 & \dots & 1 \end{bmatrix}$$

III.30 $A^{-1} = -A$.

III.31 $A^{-1}AB = O$ so $B = O$.

III.32 $A^{-1} = A$.

III.33 No.

III.34 3.

III.35 $\det A = (2a + 1)(1 - a)^2 \neq 0$.

III.36 $(\det A)^2 = \det (AA')$ and $AA' = (a^2 + b^2 + c^2 + d^2) I$.

III.37 The inverse of the associate is

$$C^{-1} = \frac{1}{\det A} A .$$

Exercises to Chapter Four (page 133)

IV.1 $(-2, 2, 1)$.

IV.2 a) $x_1 = 2,\ x_2 = t + 1,\ x_3 = t;\ t \in R$.
b) $x_1 = 0,\ x_2 = \frac{1}{2},\ x_3 = 0$.
c) no solution.

V .3 a) no solution.

b) $x = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} -5 \\ 7 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} -6 \\ 7 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$

c) no.

IV.4

$$x = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ 0 \end{bmatrix} + (2r+1) \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 1 \end{bmatrix},$$

for every integer r.

IV.5

$$x = n \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

for every integer n.

IV.6 $x = 0$.

IV.7

$$x^1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad x^2 = \begin{bmatrix} 1/5 \\ 2/5 \\ 0 \end{bmatrix}, \qquad x^3 = \begin{bmatrix} -1/5 \\ -2/5 \\ 0 \end{bmatrix}.$$

IV.8

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 3 & -5 & 6 & 4 \\ 2 & 5 & -6 & -4 \\ -1 & 5 & -7 & -3 \\ -1 & 0 & 3 & 2 \end{bmatrix}.$$

IV. 9 $a = 43$.

IV.10 $a = -2$ and $b = -3$.

IV.11 a) $c \neq -2$ and $c \neq 1$.
b) $c = 1$
c) $c = -2$.

IV.12 $c = 0$.

IV.13 $u = -74$ and $v = -76$.

IV.14 No.

IV.15 If A is regular then no.
If A is not regular then yes.

IV.16 6.

IV.17 -3.

IV.18 Hint: Apply Theorem 16.5.

IV.19 If $\gamma > 0$ then 3γ.
If $\gamma < 0$ then γ.

Exercises to Chapter Five (page 158)

V. 1 a) yes; b) no; c) yes; d) yes; e) yes.

V. 2 a) no; b) yes; c) yes; d) yes; e) yes; f) yes; g) yes.

V. 3 Hint: check that every condition in Definition 18.3 is satisfied. The sum of two functions f_1 and f_2 is the function $f_1 + f_2$ for which $(f_1 + f_2)(x) = f_1(x) + f_2(x)$ holds for all $x \in [0,1]$. Similarly the product $f_1 f_2$ is the function for which $(f_1 f_2)(x) = f_1(x) f_2(x)$. The zero element is the function that is identically zero.

V. 4 The zero element is 3^0. $\circ$ is commutative and associative. $\ast$ is associative and distributive over $\circ$:

$$(3^n \circ 3^m) \ast 3^k = 3^{(n+m)k}$$
$$3^{nk} \circ 3^{mk} = 3^{(n+m)k}.$$

V. 5 Yes.

V. 6 Hint: Show that the ring axioms are valid in this set. The multiplicative inverse of $x + y\sqrt{p}$ (where $x^2 + y^2 \neq 0$) is

$$\frac{x}{x^2 - py^2} - \frac{y}{x^2 - py^2}\sqrt{p} \quad \text{if} \quad x \pm y\sqrt{p} \neq 0.$$

However, from $x = \pm y\sqrt{p}$ it follows that $\frac{x}{y}$ is not rational, and thus $x + y\sqrt{p}$ is not in the set F'.

V. 7 Hint: For the 'if' part check that every condition in Definition 17.5 is satisfied. The 'only if' part is obvious.

V. 8 a), b).

V. 9 $[x_1, x_2] + [0, 0] \neq [x_1, x_2]$ in general.

V.10 Hint: Show that every condition in Definition 17.5 is satisfied in M, and that the mn different matrices each having $(mn - 1)$ zero elements and a single element equal to 1 form a base.

V.11 c), d).

V.12 $$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \quad \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \quad \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \quad \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}.$$

V.13 c), d).

V.14 No. $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ is not the zero matrix and still the square of its norm is

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} = \begin{vmatrix}1 & 0\\ 0 & 0\end{vmatrix} = 0,$$

V.15 $x + y = k\pi$ $(k = 0, \pm 1, \pm 2, \pm \dots)$.

V.16 $\varphi_1(x) \equiv 1, \qquad \varphi_2(x) \equiv x - \frac{1}{2}, \qquad \varphi_3(x) \equiv x^2 - x + \frac{1}{6}.$

V.17 Basic trigonometric formulae and rules of integration are to be used.

V.18 a) yes;

b) $\mathbf{x} = \frac{1}{\sqrt{3}} \tilde{\mathbf{e}}_1 + \sqrt{2}\, \tilde{\mathbf{e}}_2 + \frac{2}{\sqrt{6}} \tilde{\mathbf{e}}_3;$

c) $||\mathbf{x}|| = 3.$

V.19 $\mathbf{x}_1 = \frac{1}{3}(\tilde{\mathbf{e}}_1 + \tilde{\mathbf{e}}_2), \qquad \mathbf{x}_2 = \frac{1}{3}(-4\tilde{\mathbf{e}}_1 + 5\tilde{\mathbf{e}}_2);\ \mathbf{x}_1 \cdot \mathbf{x}_2 = 1.$

V.20 $\tilde{\mathbf{e}}_i = 2(\mathbf{v}^0 \cdot \mathbf{e}_i)\,\mathbf{v}^0 - \mathbf{e}_i$ $(i = 1, 2, 3)$
where $\mathbf{v}^0$ is the unit vector of $\mathbf{v}$.

$$T = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}.$$

Exercises to Chapter Six (page 188)

VI.1 a) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = A^{-1}$

b) $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = A^{-1}$

c) $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = A^{-1}$

d) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = A^{-1}.$

VI.2 It is BA where

$$\mathbf{A} \simeq A = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix},$$

$$\mathbf{B} \simeq B = \begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix}.$$

VI.3 $B = \begin{bmatrix} \cos\frac{2\pi}{3}, & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3}, & \cos\frac{2\pi}{3} \end{bmatrix}$ rotates,

$$C = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

$$A = C \cdot B.$$

VI.4 $A_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $B_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$,

$B_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

VI.5 $\tilde{A}_1 = \begin{bmatrix} \frac{2}{3} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & -\frac{2}{\sqrt{6}} & \frac{1}{3} \end{bmatrix}$

$\tilde{B}_1 = \begin{bmatrix} -\frac{2}{3} & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{2}{\sqrt{3}\sqrt{6}} & -\frac{2}{\sqrt{6}} & -\frac{1}{3} \end{bmatrix}$.

VI.6 Hint: the eigenvalues of A^2 are the squares of the eigenvalues of A.

VI.7 a) Hint: the eigenvalues of A^m are the mth powers of the eigenvalues of A.
b) It is $s = \text{col}\,[s_1, 0, 0, \ldots, 0]$, $s_1 \neq 0$.

VI.8 a) $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = -2$

$s_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $s_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$, $s_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

b) $\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = 3$

$$s_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \qquad s_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}.$$

c) $\lambda_1 = 1, \quad \lambda_2 = \lambda_3 = 2$

$$s_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \qquad s_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad s_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

d) $\lambda_1 = 1, \quad \lambda_2 = i, \quad \lambda_3 = -i$

$$s_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad s_2 = \begin{bmatrix} 1 - i \\ -i \\ 1 \end{bmatrix} \qquad s_3 = \begin{bmatrix} 1 + i \\ i \\ 1 \end{bmatrix}.$$

VI. 9 A one-dimensional subspace is spanned by $\mathbf{e}_1 + \mathbf{e}_2$. A two-dimensional subspace is spanned by $\mathbf{e}_3$ and $-3\mathbf{e}_1 \pm \mathbf{e}_2$.

VI.10 If $\mathbf{A} \simeq A$ and $Ax = \lambda x$, then $x^*A^* = \lambda x^*$. Multiply the first equation from the left by x^*, the second equation from the right by x, and subtract them.

VI.11 Hint: $\mathbf{A} = -\mathbf{A}^*$. Use the method of Exercise VI. 10.

VI.12
$$[\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4] \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 + \sqrt{2} & 0 \\ 0 & 0 & 0 & 2 - \sqrt{2} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix}$$

$$= 4\tilde{x}_1^2 + 2\tilde{x}_2^2 + (2 + \sqrt{2})\tilde{x}_3^2 + (2 - \sqrt{2})\tilde{x}_4^2.$$

VI.13 Hint: Show that if $\mathbf{A}$ is regular then all its eigenvalues are different from zero. Show that if $\mathbf{As} = \lambda\mathbf{s}$ then $\mathbf{As} = \dfrac{1}{\lambda}\mathbf{s}$.

VI.14 Hint: Show that if $\mathbf{As} = \lambda\mathbf{s}$ then $\mathbf{As} = -\dfrac{1}{\lambda}\mathbf{s}$ (cf. Exercise VI.13).

VI.15 a) $M_1 = \begin{bmatrix} \dfrac{3}{\sqrt{18}} & \dfrac{2}{\sqrt{8}} \\ \dfrac{3}{\sqrt{18}} & -\dfrac{2}{\sqrt{8}} \end{bmatrix}, \qquad S_1 = \begin{bmatrix} \sqrt{18} & 0 \\ 0 & \sqrt{8} \end{bmatrix}.$

b) $S = \begin{bmatrix} \dfrac{\sqrt{8} + \sqrt{18}}{2} & \dfrac{\sqrt{18} - \sqrt{8}}{2} \\ \dfrac{\sqrt{18} - \sqrt{8}}{2} & \dfrac{\sqrt{8} + \sqrt{18}}{2} \end{bmatrix}, \quad M = \dfrac{1}{2}\begin{bmatrix} \dfrac{6}{\sqrt{18}} & \dfrac{4}{\sqrt{8}} \\ \dfrac{6}{\sqrt{18}} & -\dfrac{4}{\sqrt{8}} \end{bmatrix}$

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